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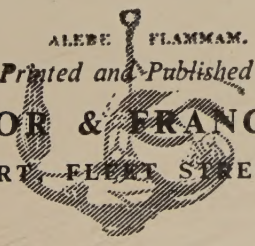

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CII. *Gamma Radiation from the Decay of* $^{238}Pu_{94}$, $^{242}Cm_{96}$ *and* $^{243}Cm_{96}$

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[Received February 10, 1956]

Abstract

Accurate energy measurements on the gamma rays from the decay of ^{238}Pu and ^{242}Cm have been made with proportional counters. Values of $43{\cdot}49\pm0{\cdot}08$, and $99{\cdot}8\pm0{\cdot}4$ and $153{\cdot}1\pm0{\cdot}6$ kev for ^{238}Pu, and $44{\cdot}03\pm0{\cdot}06$, $101{\cdot}80\pm0{\cdot}17$ and $157{\cdot}61\pm0{\cdot}3$ kev for ^{242}Cm were obtained. The ratios of these energies for each nuclide agree well with those predicted in the strong coupling limit of the Unified model. Gamma rays having energies of $210\pm1{\cdot}5$, 228 ± 2 and 277 ± 2 kev and K–X radiation were found from the decay of ^{243}Cm. These gamma rays are shown to be magnetic dipole with a 30% mixture of electric quadrupole ; the magnetic dipole component is 10^4 times slower than the single particle estimate.

§ 1. Introduction

This work was originally undertaken with the aim of investigating the modes of decay of the alpha particle emitters ^{238}Pu and ^{242}Cm. At that time (early 1953) there was little evidence regarding the second and higher excited states of the heavy even–even nuclei and their systematic variation with mass number was not realized. It was apparent from our early measurements that the gamma ray spectra from the decay of ^{238}Pu and ^{242}Cm were similar. Later it was found that the ratios of the energies of these gamma rays fitted very well with the values to be expected if they arose from the decay of nuclear rotational states as predicted by Bohr and Mottelson (1953) from the unified model, and as demonstrated, with rather less precision, by the work of Asaro and Perlman (1954) and Asaro, Thompson and Perlman (1953) on the alpha-particle fine structure of the even–even elements.

In view of the rather precise predictions of the unified model regarding the relative spacings of energy levels it seemed desirable to make the energy measurements as accurate as possible. Krypton and xenon filled proportional counters of the type previously described by West (1953) were used. The main advantage of these counters is their resolution, approximately 5% full width at half height for gamma ray energies of 100 kev. Their intrinsic efficiency for detection is also quite good, being about 1·5% for radiation of 100 kev in a 2 in. diameter counter filled to atmospheric pressure with xenon.

† Communicated by the Authors.

A preliminary account of the work on ^{238}Pu and ^{242}Cm was presented at the Birmingham Nuclear Physics conference in July 1953 (Rose and Newton 1953) and the later results have also been quoted in other publications (Bohr 1954, Newton 1954). The energy values given in this paper differ slightly from those previously reported because the energy of the gamma ray from the decay of ^{241}Am, which was used as a standard for these measurements, has recently been revised. It was originally reported as 59·78±0·02 kev (Browne 1952, Asaro, Reynolds and Perlman 1952) but is now given as 59·568±0·017 kev (Day 1955) and 59·62± 0·06 kev (Jaffe *et al.* 1955).

§ 2. Preparation of ^{242}Cm and ^{238}Pu Sources

Curium 242 was prepared by the neutron irradiation of americium 241 :

$$\text{Am 241 (n,}\gamma\text{) Am 242 m} \xrightarrow[16\text{hr}]{\beta} \text{Cm 242.}$$

Two samples of curium were used in the present work, one from an irradiation in BEPO at Harwell, and the other from a much more intense irradiation in the NRX reactor at Chalk River. The integrated neutron doses for these irradiations were approximately 2×10^{18} and 10^{20} neutrons/cm^2, respectively. It is unlikely that an appreciable amount of Cm 243 was formed in the BEPO irradiation, but a fractional percentage by weight of this isotope could have been formed in the NRX irradiation, by further neutron capture in Cm 242.

In each case, the curium was separated from the irradiated americium by the following chemical steps :

(*a*) Precipitation as fluoride (with a little residual trivalent americium) from a solution in which most of the americium had been oxidized to the fluoride-soluble hexavalent state. The oxidizing agent used was ammonium persulphate with silver catalyst, and the technique was similar to that described by Asphrey *et al.* (1951). This step also separated the curium from most fission products except the rare earths.

(*b*) Separation from rare earths by elution from cation exchange column with 12 N hydrochloric acid (Street and Seaborg 1950).

(*c*) Separation of curium from residual americium by elution from a cation exchange column with ammonium citrate solution at 85°C. This was for several years the standard separation technique for the higher actinide elements, and it is described (*inter alia*) by Thompson *et al.* (1950). A two-stage separation was necessary to achieve the desired purity. The early curium-rich fractions from the first elution were combined and a second elution from a smaller column was carried out.

Plutonium 238, the α-decay daughter of 162 day curium 242 was extracted from purified curium fractions which had been standing for several months, by a further ion exchange step. The plutonium fraction was purified chemically by conventional solvent extraction and precipitation reactions on an ultramicro scale. The Pu 238 used in the present work was shown by pulse analysis to contain negligible amounts of Cm 242 and other α-emitters.

Sources on aluminium foils were prepared by the tetra-ethylene glycol spreading technique (Glover and Borrell 1955) and were assayed by direct low geometry α-counting.

The larger sources, in sealed tubes, were prepared by evaporating a known volume of solution to dryness or to a small volume. The solution had previously been assayed by taking aliquots with a micro-pipette and α-counting in the low geometry counter.

§ 3. Experimental Arrangement

A diagram of the apparatus is shown in fig. 1. The glass proportional counter A, of the Maze type, was ~20 in. long and had an inside diameter

Fig. 1

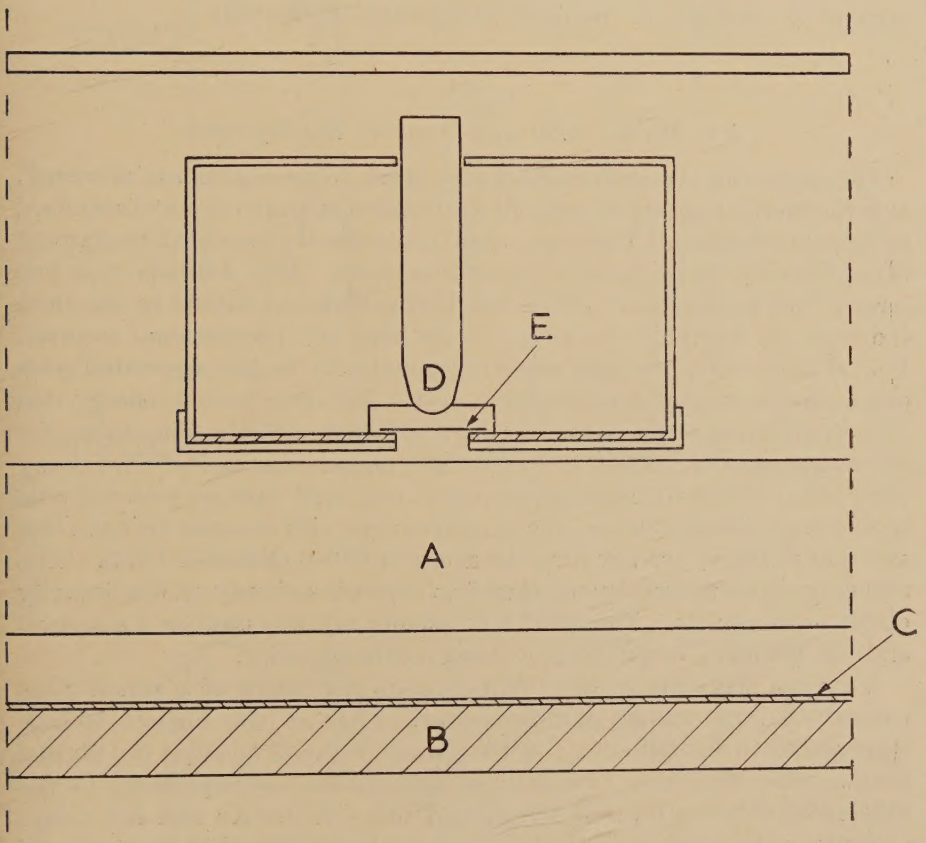


Diagram of the apparatus.

of 2 in. and a 0·005 in. tungsten wire. The counter A was placed in the lead castle B in order to reduce the general background. Lead K radiation (~75 kev) is produced by the process of fluorescent excitation from gamma radiation having an energy greater than the K-binding energy of lead. For this reason the castle was usually lined with 1 mm

cadmium sheet C, which has a high attenuation for 75 kev radiation. The fluorescent radiation from the cadmium has an energy of 26 kev. This, being in general below the energy region of interest, was not troublesome.

The sources D were placed as shown in a box made from 2 mm cadmium sheet and lined inside at the bottom with lead. This box had the following functions : (*a*) To provide some collimation so that absorbers E could be used, (*b*) to give a reproducible position for the sources, (*c*) to minimize the amount of backscattered radiation detected in the counter. The backscattered Compton radiation is troublesome at these low energies because it differs in energy from the primary radiation by only a small amount, e.g. by 11·4 kev for primary energy of 60 kev.

The counter was sometimes used in a longitudinal magnetic field. This field, which could be varied from zero to 8000 gauss, was produced by the magnet of the large electromagnetic separator at Harwell.

§ 4

4.1. *Measurement of the Gamma Ray Energies*

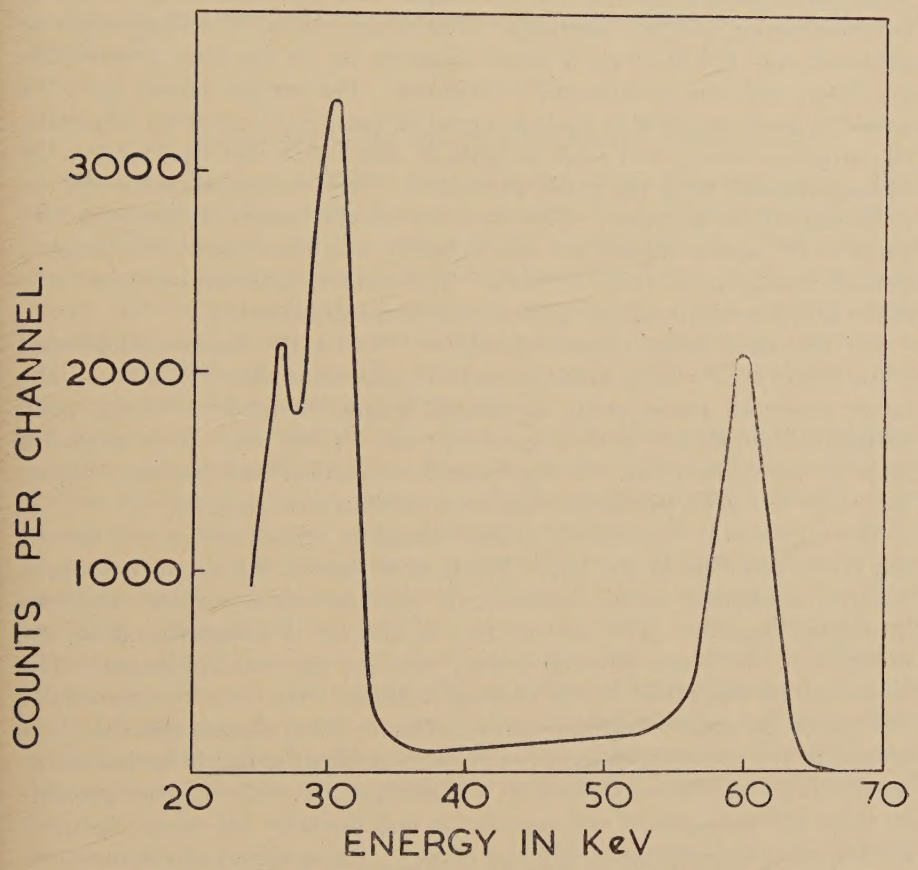
The proportional counter cannot give absolute measurements of energy. It is therefore necessary to refer all the measurements to those of standard calibration energies. These are most conveniently provided by gamma rays of known energy from radioactive sources. It is desirable that the error in the measurement of the standard gamma ray should be less than the error in the relative measurements with the proportional counter. It is also necessary that the standard gamma ray be not associated with other gamma rays of comparable intensity and close to it in energy and that the lifetime of the source is long compared with the time taken for the measurements. The $59{\cdot}57 \pm 0{\cdot}02$ kev gamma ray from ^{241}Am decay (Day 1955) fulfills all these requirements very well, but we were not able to find any suitable sources giving gamma rays with energies between this and that of $191{\cdot}5 \pm 0{\cdot}5$ kev from the decay of ^{114m}In (Mihelich 1952). The accuracy of our measurements therefore depends seriously on the linearity of our arrangement. The 59·57 kev gamma ray was used as a standard and the 191·5 kev radiation as a check on the linearity.

A typical spectrum of the ^{241}Am gamma ray taken in a xenon filled counter (fig. 2) consists of three peaks. That at the highest voltage corresponds to the full energy of the gamma ray and is called the normal peak, whilst the other two are ' escape ' peaks, corresponding to the differences between the full energy and those of the Kα and Kβ x-rays of xenon ; the energies of these x-rays can be estimated to be 29·66 and 33·61 kev from the table of x-ray absorption edges given by Hill *et al.* (1952). In this particular case the Kβ escape peak contains also a contribution from the 26·4 kev gamma ray of ^{241}Am.

For gamma ray energies greater than about 60 kev the Kα and Kβ peaks are no longer resolved. In this case the difference in energy between

the maxima of the full energy and escape peaks is a function of the resolution of the counter for the particular energies concerned. A correction for this is easy to make since the ratio of intensities of the Kα and Kβ peaks is constant and the variation of counter resolution with energy can be found empirically. The corrections were 0·01 kev and 0·64 kev for escape peaks having energies of 60 kev and 160 kev respectively.

Fig. 2



Pulse height distribution from a 59·6 kev gamma ray taken in a xenon filled proportional counter.

Both the normal and escape peaks were used in the energy measurements. As the escape peaks are not single lines and therefore do not have a Gaussian shape, it is not possible to use the method of the Ogive plot for determining their position. We therefore defined a peak position, for the purpose of these measurements, as the intersection with the curve of its median line.

Energy measurements of the type described here suffer from errors due to the following : (*a*) non-linearity in the electronic apparatus, (*b*) non-linearity in the proportional counter, (*c*) instability in the counter power supplies, in the amplifier and the kicksorter, and (*d*) statistical fluctuations.

In the original measurements a thirty-channel pulse amplitude analyser (A.E.R.E. type 1091A) was used and set up to be linear, with an accuracy of one part in 10^4, by means of a calibrated pulse generator (A.E.R.E. type 1091A). The linearity of the amplifier was checked as follows. A condenser was charged from the positive d.c. output of the type 1091A pulse generator and discharged twice per second by means of two oscillating mercury switches. The sharp negative voltage step so obtained was fed through a small capacity on to the first grid of the amplifier, with the counter still connected. The output pulses from the amplifier were arranged to have a spread of about $\pm 0{\cdot}5$ volts by adjusting the amplifier noise, and their amplitude was controlled by varying the voltage derived from the pulse generator ; they were observed with the pulse amplitude analyser. The accuracy of the measurements was two parts in 10^3 and no significant non-linearity was observed in the range of output voltage from 16 to 56 volts. The error in this measurement was taken into account in assessing the errors of the final results.

For the more recent measurements on ^{242}Cm a 100 channel kicksorter (type 1363A) and a type 1405A waveform generator were available. The latter produced pulses with adjustable height (defined to 0·1%), with adjustable length and with a repetition rate of 200 c/s ; they were fed into the amplifier input. It was possible to achieve an accuracy higher than that of the earlier measurements with this arrangement.

The linearity of a proportional gas counter is dependent on two factors (cf. West 1953, Chs. II and IV). The first of these is the magnitude of the electron avalanche in the counter ; if this exceeds a certain size then saturation sets in. The size of the avalanche is proportional to the product of the recoil electron energy and the gas mutliplication. The latter can be controlled by the counter voltage, which was therefore kept well below the point where saturation set in for these measurements. The second factor upon which the linearity is dependent is the variation of the mean energy per liberated electron with energy. It did not prove possible to show the presence of any significant non-linearity for recoil electrons having energies between 11 kev and 60 kev ; the accuracy of the measurements varied from 7 parts in 10^3 at 11·5 kev to 2 parts in 10^3 for 60 kev. Sources of ^{153}Gd and ^{241}Am respectively were used in these measurements. The linearity in the region of energy greater than 60 kev was checked by making a measurement of the energy of the gamma ray from the decay of ^{114m}In ; values of $191{\cdot}3 \pm 0{\cdot}7$ kev in the earlier measurements and $191{\cdot}3 \pm 0{\cdot}4$ kev in the later measurements were obtained. These are in good agreement with the best reported value of $191{\cdot}5 \pm 0{\cdot}5$ kev (Mihelich 1952) obtained from measurements on the conversion lines.

The stability in channel width of the pulse amplitude analyser during a measurement was better than 1%, and the stability of a given channel

threshold about $\pm 0{\cdot}025$ volts. Since the pulse heights normally used were between 30 and 60 volts, errors due to drifts in the pulse amplitude analyser were negligible compared with those from other sources. Instability in the counter power supplies and in the amplifier caused an overall long term variation of about $\pm 1\%$. The line to be measured and the calibration line were always observed simultaneously so that errors due to small changes in the overall gain of the system were eliminated. When the gamma ray under investigation had an energy greater than 100 kev it was not compared directly with the standard ^{241}Am line but with another line close to it in energy ; this substandard line was then compared directly with the ^{241}Am line or with another intermediate substandard.

4.2. *Estimation of the Errors of the Energy Measurements*

The error in peak position was estimated in the following way. First a ' visual error ' was defined as one quarter of the difference between two visually estimated extreme values of the peak position. This error was determined by two observers to reduce the subjectivity of the determination, and in general good agreement was found. It is related to the statistical error in the measurements, for much more extreme curves can be drawn through a set of points with poor statistical accuracy than through a similar but more accurate set.

In order to relate the visual error to the standard error of the measurements, the standard deviation from the mean of fourteen determinations of the energy of a 44 kev gamma ray was compared with that calculated from the visual errors. The difference between the two results is not significant. The standard error of the energy was determined by suitably combining the visual error with that of the linearity measurement. When the old value of $59{\cdot}78 \pm 0{\cdot}02$ kev (Browne 1952, Asaro, Reynolds and Perlman 1952) was used as standard there was a ' barely significant ' difference between the energy values obtained by comparison with the normal and escape peaks of the ^{241}Am gamma ray and in this case the visual error appeared to be a factor of 1·4 times too small. This difference disappeared when the more recent value of 59·57 kev (Day 1955) was used.

§ 5. Measurement of the Intensities

With the experimental arrangements used here, it is possible to deduce the relative intensities of gamma rays having energies less than 50 kev from the areas under their respective peaks and a knowledge of their absorption coefficients in the counter gas and in the absorbers.

The mass absorption coefficients for photoelectric absorption in krypton and xenon were obtained as follows. Values for the total mass absorption coefficients were obtained from the tables given by Compton and Allison (1935) ; they had to be obtained by interpolation for xenon. After correction for absorption by the Compton process they were plotted

on a log–log scale against the gamma ray energies. The points lay on straight lines so that the photoelectric absorption coefficients could be read off for any energy.

When the ranges of the recoil electrons become comparable with the counter radius a fraction of the pulses is lost from the peak by wall collisions. In general these ' lost ' pulses have an energy much below the peak value (West *et al.* 1952). The loss of electrons can be reduced by placing the counter in a longitudinal magnetic field (West and Rothwell 1950). The electron tracks are then curled up around the field direction.

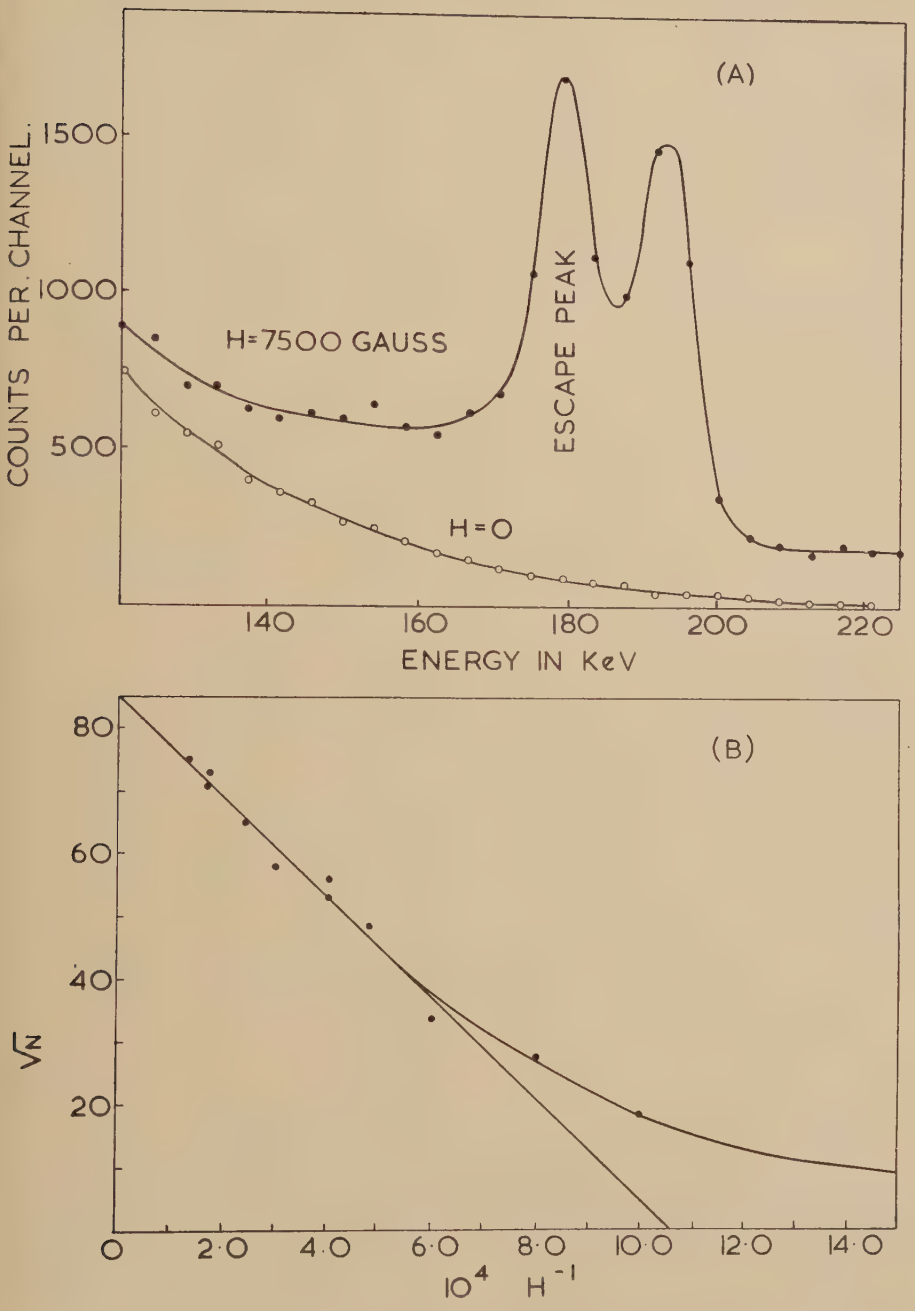
If an infinitely large longitudinal magnetic field was applied to a long proportional counter, irradiated with gamma rays in its central region only, then there would be no loss of electrons provided that they could not escape out of the ends. Hence, if the law relating the counter efficiency to the magnetic field were known, it would be possible to calculate the efficiency for any field. A rigorous theoretical calculation of this law would be difficult since the effect of scattering, which is considerable for low energy electrons, would have to be taken into account. Elementary considerations, neglecting the effect of scattering, suggest that the efficiency $\epsilon\ (H)$ of the counter for magnetic field H, and energy E, should be given approximately by an expression of the form $\epsilon(H)= [\epsilon(\infty)-\epsilon(0)](1-\beta/H)^2$ where β is a constant for a given E and where H is so large that the radius of curvature of the electron is smaller than the counter radius. From our experiments it appeared that an expression of this form was in fact obeyed. In figs. 3 A and 4 A are shown the spectra of the 192 kev gamma ray from ^{114}In taken in a krypton filled counter and of the 411 kev gamma ray from ^{197}Au taken in a xenon counter with and without a magnetic field, whilst in figs. 3 B and 4 B are plotted the square roots of the numbers of counts in the peaks against the inverse of the magnetic fields. Similar results were obtained for gamma rays of other energies.

It seems plausible that this semi-empirical relation should be obeyed to infinite field. Relative intensities were therefore estimated by obtaining a few points on the linear portion of the curve and extrapolating to infinite field.

The results were compared with those obtained with a sodium iodide scintillation detector and good agreement was found. The procedure with the scintillation counter is direct since there is no significant loss for low gamma ray energies provided that the escape peak is included ; however the resolution is worse than that of a proportional counter by approximately a factor of five. The rather complex procedure for obtaining relative intensities with a proportional counter is therefore justified when a complicated spectrum is being analysed.

The numbers of low energy ($\sim$44 kev) gamma rays per alpha-particle disintegration were determined as follows. The intensity of the gamma rays from a source of each isotope was measured relative to that of the 59·57 kev gamma rays from an ^{241}Am source by counting under the same geometrical conditions and correcting for counter efficiencies. By

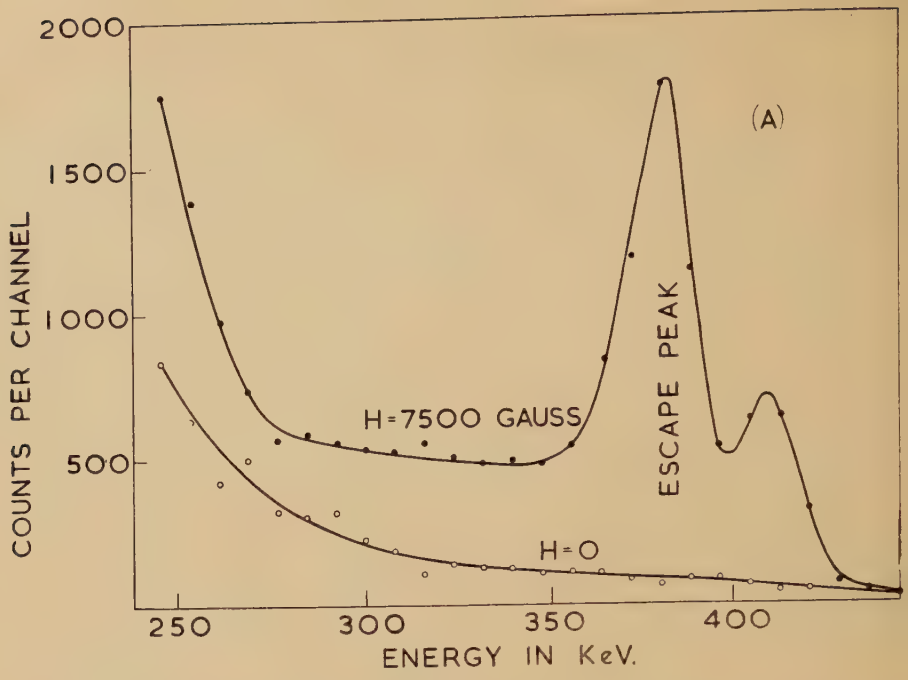
Fig. 3

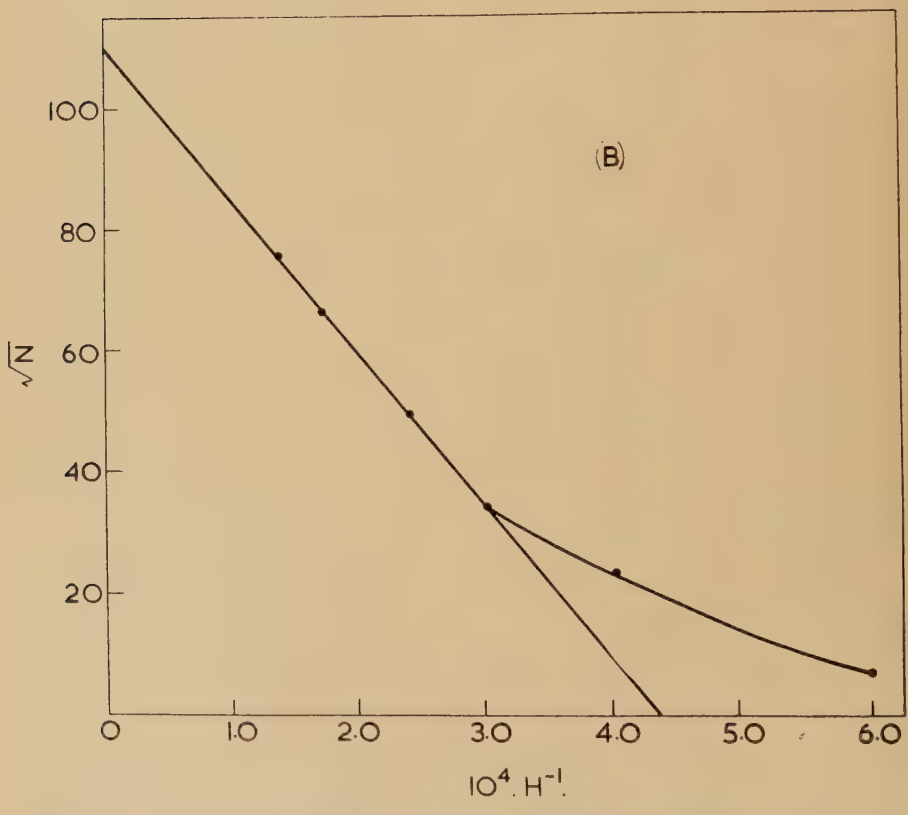


192 kev gamma ray and a krypton filled counter.

(*a*) **Pulse height distributions with and without a longitudinal magnetic field** H.

(*b*) **Plot of the square root of the number** N **of counts in the peak as a function of** H^{-1}.





411 kev gamma ray and a xenon filled counter.

(*a*) Pulse height distributions with and without a longitudinal magnetic field H.

(*b*) Plot of $\sqrt{N}$ as a function of H^{-1}.

measuring the disintegration rates of the sources and assuming the value of $0{\cdot}40 \pm 0{\cdot}025$ for the number of 59·57 kev gamma rays per alpha-particle (Beling *et al.* 1952) the number of 44 kev gamma rays per disintegration was obtained. The ^{242}Cm and ^{241}Am sources were thin and their disintegration rates for alpha-particle emission were measured in a standard low geometry counter. The ^{238}Pu was in solution and contained in a thin walled polythene vessel. Its source strength was found by alpha-counting a known weight of solution.

§ 6. Experimental Results

6.1. *The Gamma Radiation from the Decay of* $^{238}Pu_{94}$

The energies and relative intensities of the gamma ray lines are shown in table 1.

Table 1. Energies and relative intensities of gamma rays from the decay of $^{238}Pu_{94}$

Energy of line in kev	$43{\cdot}49 \pm 0{\cdot}08$	$99{\cdot}82 \pm 0{\cdot}4$	$153{\cdot}1 \pm 0{\cdot}6$
Relative intensity	1·00	$(2{\cdot}75 \pm 0{\cdot}25)\ 10^{-1}$	$(3{\cdot}0 \pm 0{\cdot}4)\ 10^{-2}$

Two sources of ^{238}Pu, which had been put through different purification procedures, were used in these measurements. They both gave the lines reported in the table with the same relative intensities. Some weak high energy radiation (~750 kev) was also observed but this differed in intensity in the two sources and may have been due to fission product contamination. It was not possible to establish definitely the presence of any other radiation, apart from the *L* radiation of U. In particular an upper limit to the intensity of uranium-*K* *X*-radiation relative to the 43 kev radiation was 3×10^{-2}.

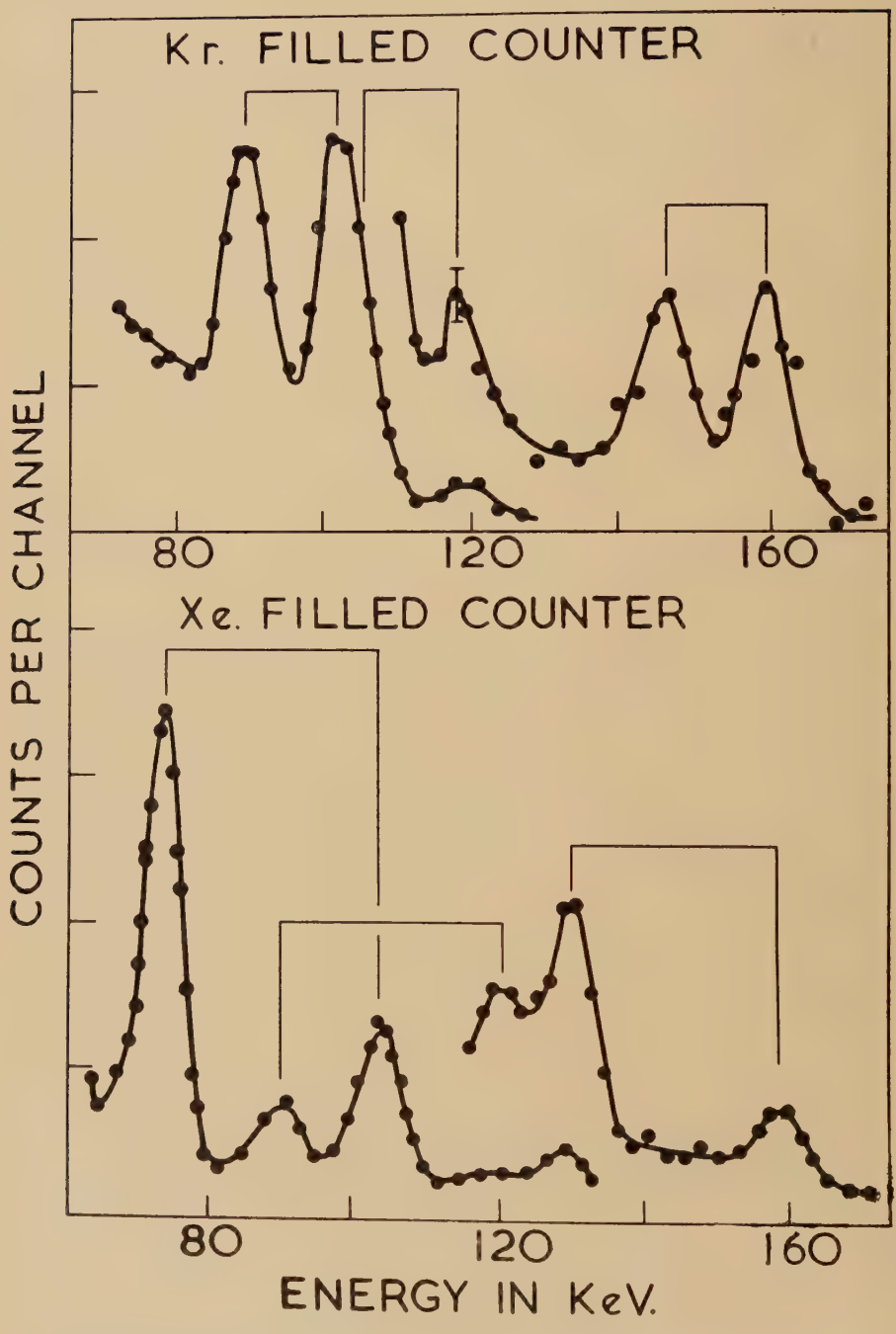
The absolute intensity of the 43 kev radiation was found to be $(3{\cdot}8 \pm 0{\cdot}4) \times 10^{-4}$ gamma rays per alpha-particle.

These results are consistent with those reported by Asaro and Perlman (1954).

6.2. *Gamma Radiation from the Decay of* ^{242}Cm and ^{243}Cm

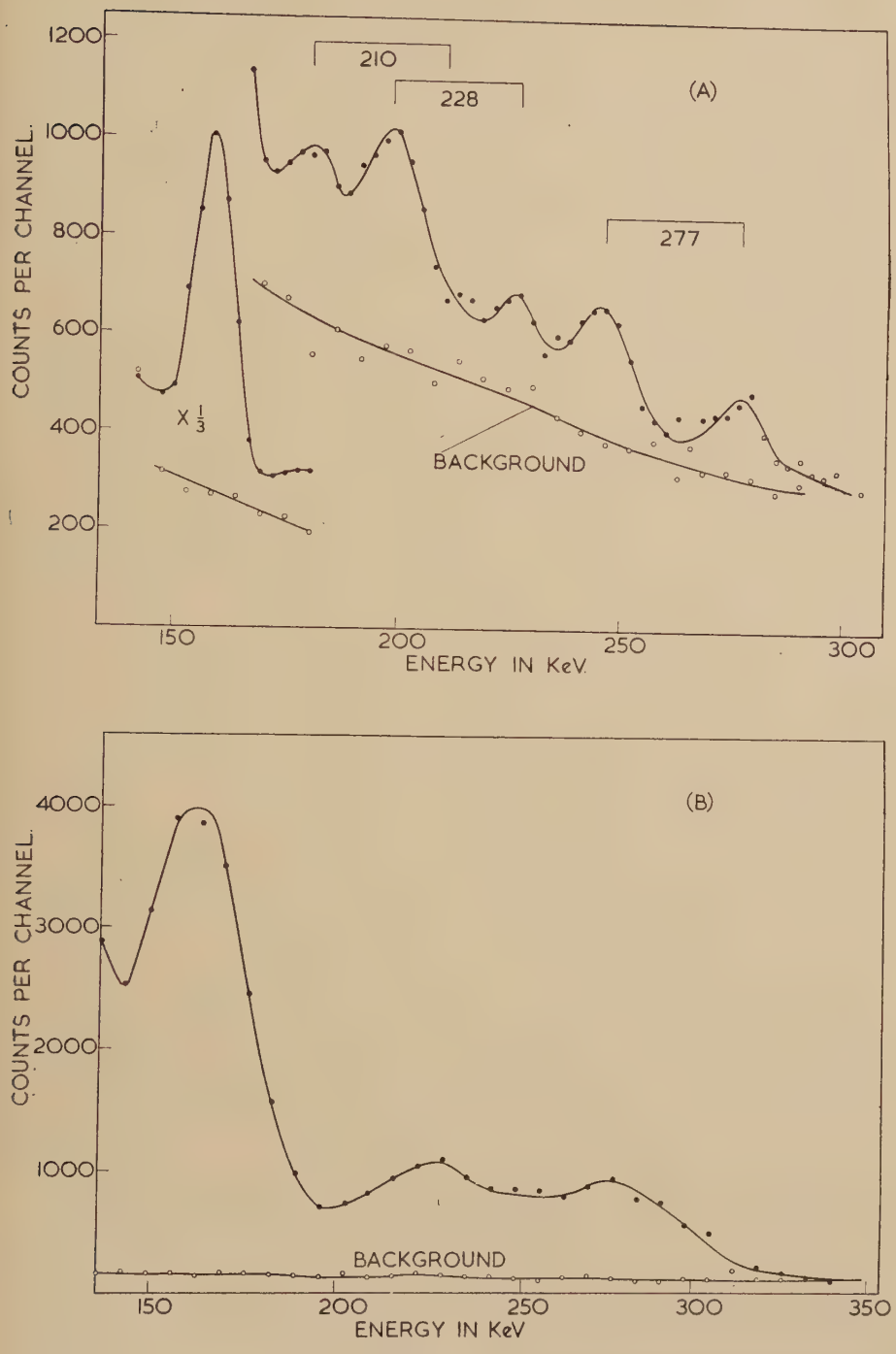
As explained in § 2, sources of ^{242}Cm in general contain some ^{243}Cm; the ratio of ^{242}Cm to ^{243}Cm depends on the irradiation of the source and the time after preparation. Sources having different irradiation histories were studied and the decay of one source containing an appreciable fraction of ^{243}Cm gamma activity was followed over a period of 780 days. From these measurements and from known data on the beta decay of ^{239}Np, which leads to the same daughter nucleus as does ^{243}Cm it was possible to assign the various observed lines to the decay of ^{242}Cm, ^{243}Cm and ^{241}Am, which occurred as a contaminant in some of the sources.

Fig. 5



Pulse height spectra from Cm sources taken in krypton and xenon filled counters.

Fig. 6



Pulse height spectra in the energy range 140–320 kev from a source containing ^{242}Cm and ^{243}Cm.

(*a*) Taken in a xenon filled proportional counter with a longitudinal magnetic field of 7500 oersteds.

(*b*) Taken in a sodium iodide crystal.

The spectra from the gamma rays with energies between 60 and 170 kev taken in xenon and krypton filled proportional counters are shown and compared in fig. 5 ; the normal and escape peaks of the various lines are joined by bars. The advantage of using two counters with different gas fillings in investigation of a complex spectrum is well illustrated ; the different escape energies of the two gases make it easy to tell which are the normal and which are the escape peaks.

In fig. 6A is shown the spectrum of the gamma rays with energies greater than 150 kev taken in a xenon filled counter placed in a longitudinal magnetic field of 7500 oersteds, whilst in fig. 6B is shown the spectrum of the gamma rays taken in a sodium iodide crystal.

The peak at 180 kev in fig. 6A is likely to be the escape peak of a 210 kev gamma ray rather than the normal peak of a 180 kev gamma ray for the following reasons, (*a*) the peak at 200 kev is too large to be accounted for solely by the escape peak of a 228 kev gamma ray, (*b*) in figs. 6B and 8B the intensity of the peak at 225 kev relative to that at 275 kev is $1{\cdot}00 \pm 0{\cdot}05$. This is consistent with the peak at 225 kev arising from 210 and 228 kev lines but not consistent with it arising solely from the 228 kev line, if the relative intensities of the 228 and 277 kev lines are as given in table 3, (*c*) a gamma ray of energy 210 kev is observed in the decay of ^{239}Np (Graham and Bell 1951) and would also be expected to appear in the ^{243}Cm decay (see § 7.2).

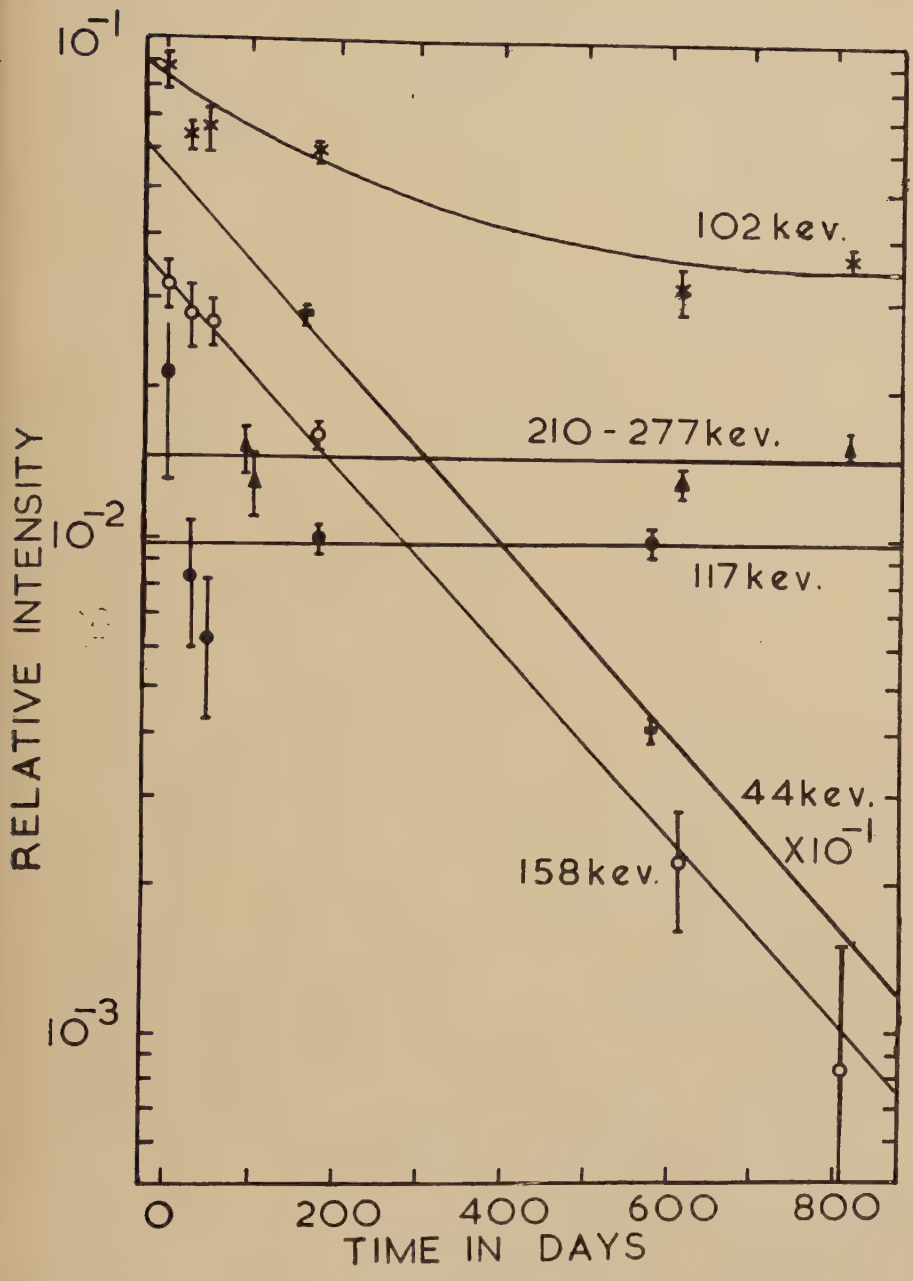
The variation of the intensities of the various lines relative to the 59·6 kev ^{241}Am line is shown in fig. 7. ^{241}Am has a half life of 470 years (Harvey 1952) so that it does not decay appreciably in the time involved here. It can be seen that the 44 kev and 158 kev lines decay with a half life consistent with that of 162·5 days for ^{242}Cm (Glover and Milsted 1954) ; the 102 kev line decays with a longer apparent half life and the 117 kev and 210–277 kev lines remain constant within the errors of the measurements. The half life of ^{243}Cm has been reported to be 35 years (Asaro 1953) so that radiation from it will remain approximately constant during the period of our observations. It therefore seems likely that the 117 kev and the 210–277 kev radiations arise from from ^{243}Cm decay.

Gamma ray spectra, taken with a sodium iodide crystal, for two sources : (*a*) with negligible ^{243}Cm gamma activity, and (*b*) containing a large fraction of ^{243}Cm gamma activity are compared in fig. 8. It can be seen that the 102 kev line must be present in both the ^{242}Cm and ^{243}Cm spectra. This explains the longer apparent half life for the 102 kev line shown in fig. 7.

In table 2 are shown the energies and relative intensities of the gamma rays which are attributed to the decay of ^{242}Cm. In addition to these the *L X*-radiation of Pu was observed. The number of 44·03 kev gamma rays per alpha-particle was found to be $(3{\cdot}9 \pm 0{\cdot}7) \times 10^{-4}$.

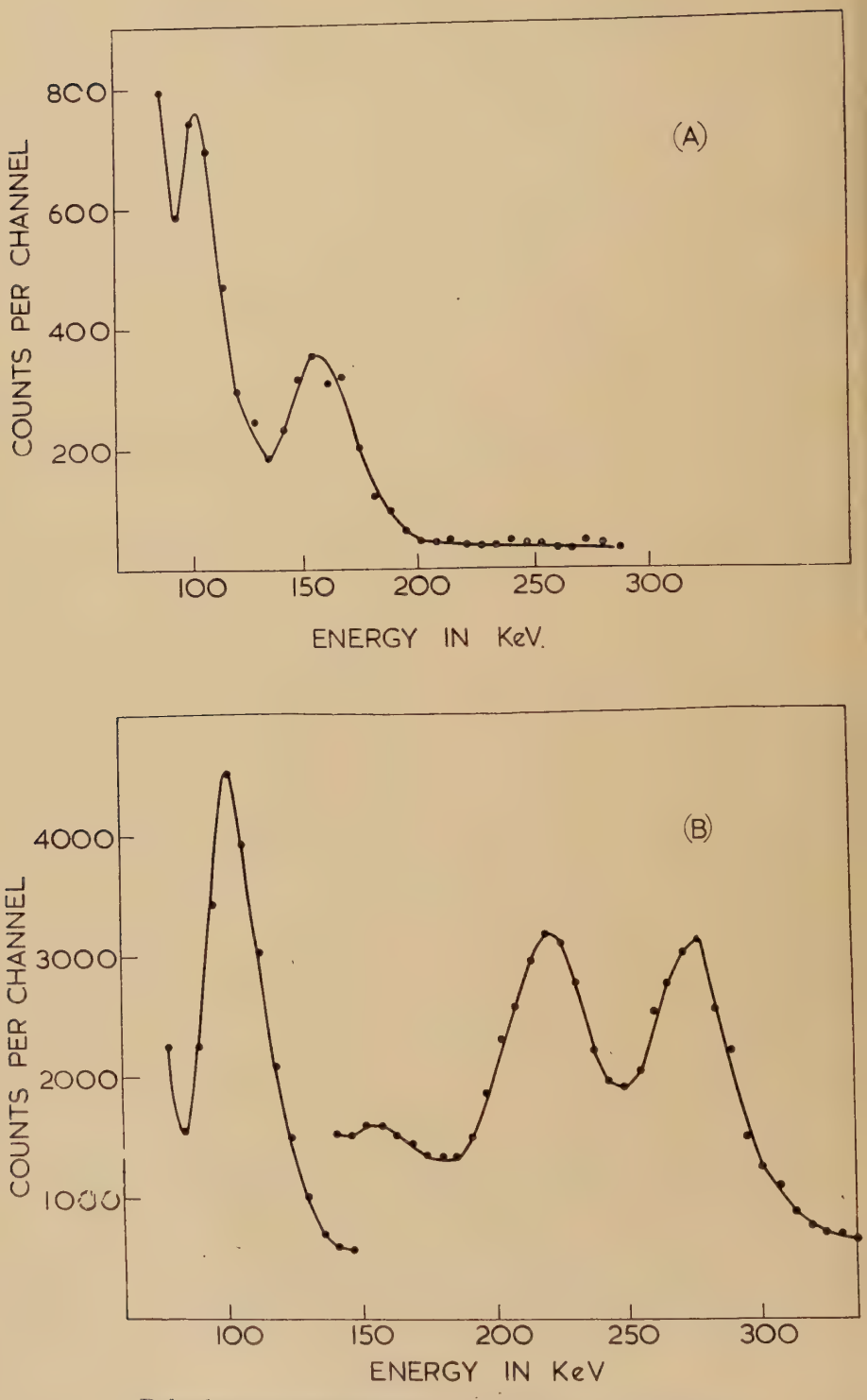
The energies and relative intensities of the lines which are attributed to the decay of ^{243}Cm are shown in table 3. The energies of the three higher energy gamma rays are in good agreement with those of 210, 227

Fig. 7



The intensities of the various lines from a Cm source, relative to that of the 59·6 kev line from ^{241}Am, as a function of time.

Fig. 8



Pulse height distribution taken in sodium iodide crystals.
(*a*) From a Cm source containing negligible ^{243}Cm activity.
(*b*) From a Cm source containing a large fraction of ^{243}Cm activity.

and 276 kev obtained by Graham and Bell (1951) from a study of the conversion lines from the decay of ^{239}Np. The two lower energy lines

Table 2. Energies and relative intensities of the gamma ray lines from the decay of $^{242}Cm_{96}$

Energy of line kev	44·03±·06	101·80±·17	157·61±·3
Relative intensity	1·00	$(9\pm1\cdot1)10^{-2}$	$(6\cdot0\pm\cdot5)10^{-2}$

have energies in good agreement with those of 102·29 and 117·96 kev predicted for the Kα and Kβ x-rays of plutonium from the tables of Hill *et al.* (1952). The intensity of the plutonium-K–X–radiation relative to that of the three gamma rays was measured with a sodium iodide crystal and found to be 2·34±0·25.

Table 3. Energies and relative intensities of the gamma rays from ^{243}Cm decay

Energy of line kev	102·15±·2	117·30±0·4	210±1·5	228±2	277±2
Relative intensity	3·6 ±·4	1·4 ±0·2	0·5 ±·15	0·65±0·15	1·00

§ 7. Interpretation

7.1. *The Gamma Radiation from the Decay of* ^{238}Pu and ^{242}Cm

The gamma ray spectra arising from the decays of ^{238}Pu and ^{242}Cm are very similar. The assignment of the gamma ray lines to transitions between excited states in ^{234}U and ^{238}Pu has been discussed by Asaro and Perlman (1954) Asaro, Thomson and Perlman (1953) and Rose and Newton (1953). The proposed decay schemes are shown in fig. 9. The two lower energy gamma rays in each case almost certainly correspond to transitions between the three lowest states, since their energies correspond, within the experimental errors, to the energy differences of these states as determined by the alpha spectra. Asaro and Perlman (1954) have shown that 153 kev and 100 kev gamma rays from ^{238}Pu decay are in coincidence. This gives strong support to the proposed decay scheme. Support is also given by the fact that the ratios of the energies of the excited states and the cascade gamma ray transitions are just as expected from the strong coupling limit of the Unified Nuclear Model of Bohr and Mottelson (1953).

According to this model, in even–even nuclei well removed from closed shells there should exist rotational levels. Their energies, to first order, are given by the expression

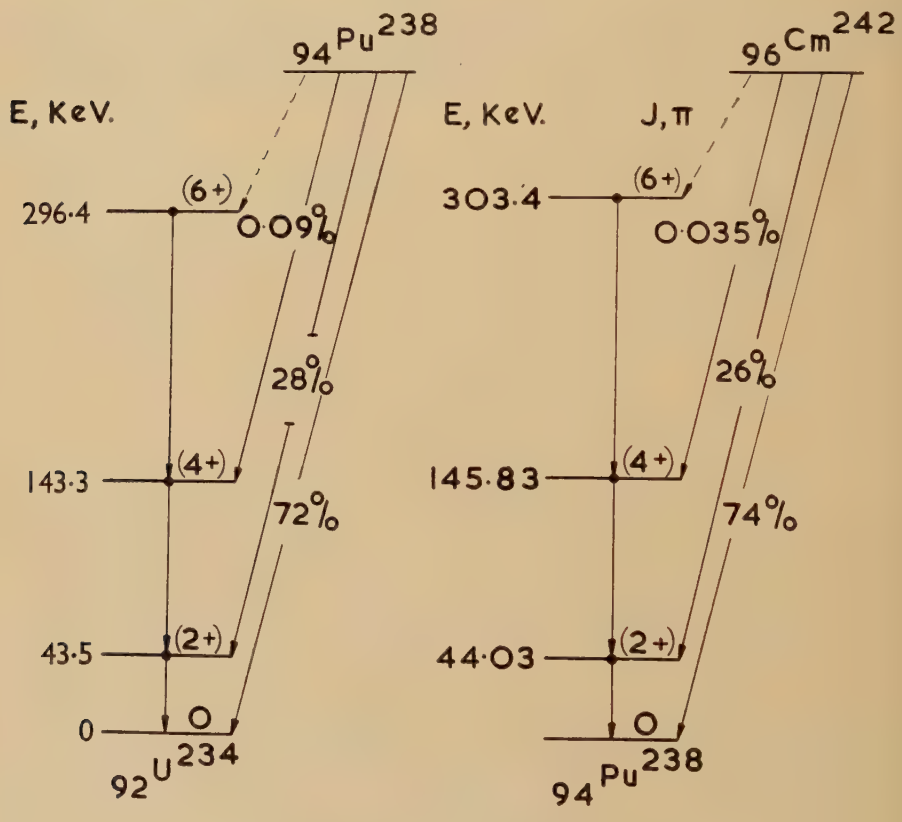
$$E(I)=\frac{\hbar^2}{2\mathfrak{I}}I(I+1),\ I=0,\ 2,\ 4 \text{ even parity}$$

where I is the spin of the level and $\mathfrak{I}$ is an effective moment of inertia related to the deformation of the nucleus from the spherical shape. The rotational levels should decay by cascade $E2$ transitions to the ground state. A correction E to the energies, analogous to that due to the rotation–vibrational interaction in molecular spectra is given by

$$E(I) \approx -2\left(\frac{1}{\hbar\omega_{\text{vib}}}\right)^2\left(\frac{\hbar^2}{\mathfrak{I}}\right)^3 I^2(I+1)^2$$

where ω_{vib} is a measure of the vibrational frequencies of the nucleus.

Fig. 9



The decay schemes of ^{238}Pu and ^{242}Cm.

In table 4 are shown the ratios of the energies E_2 and E_3 of the second and third excited states of ^{234}U and ^{238}Pu to those E_1 of their first excited states. These ratios agree remarkably well with those expected from the first order theory. The small deviations can be taken into account by assuming values for ω_{vib} of $1{\cdot}1\pm0{\cdot}1$ Mev and $1{\cdot}41\pm0{\cdot}08$ Mev for ^{234}U and ^{238}Pu respectively. These values are to be compared with a value of about 1 Mev expected on the hydrodynamic approximation (see Bohr and Mottelson 1953, p. 14).

7.2. *The Decay of* ^{234}Cm

The decay of ^{243}Cm has been investigated by Asaro, Thomson and Perlman (1953) and Asaro and Perlman (1954). They found alpha-particle groups having energies of 5·985, 5·777, 5·732 and 5·677 Mev and photons with energies of 104 kev, 226 kev and 278 kev, the latter two being in coincidence with the 5·777 Mev alpha group. A number of authors (Graham and Bell 1951, Engelkemeier and Magnusson 1955, Fulbright 1949, Slätis 1948) have studied the beta decay of ^{239}Np which leads to the same residual nucleus ^{239}Pu as is reached by the alpha decay of ^{243}Cm. The Coulomb excitation of ^{239}Pu has been studied by Newton (1955). A level scheme derived from all these measurements is shown in fig. 10.

It can be seen that the 277, 228 and 210 kev radiations all arise from the metastable state labelled 285 kev. If the decay scheme of fig. 10, is correct, the predominant part of the plutonium K–X-radiation which we observe must arise from the internal conversion of the 277, 228 and 210 kev transitions.

Table 4. Comparison of experimental and theoretical energy ratios for the excited states of ^{234}U and ^{238}Pu

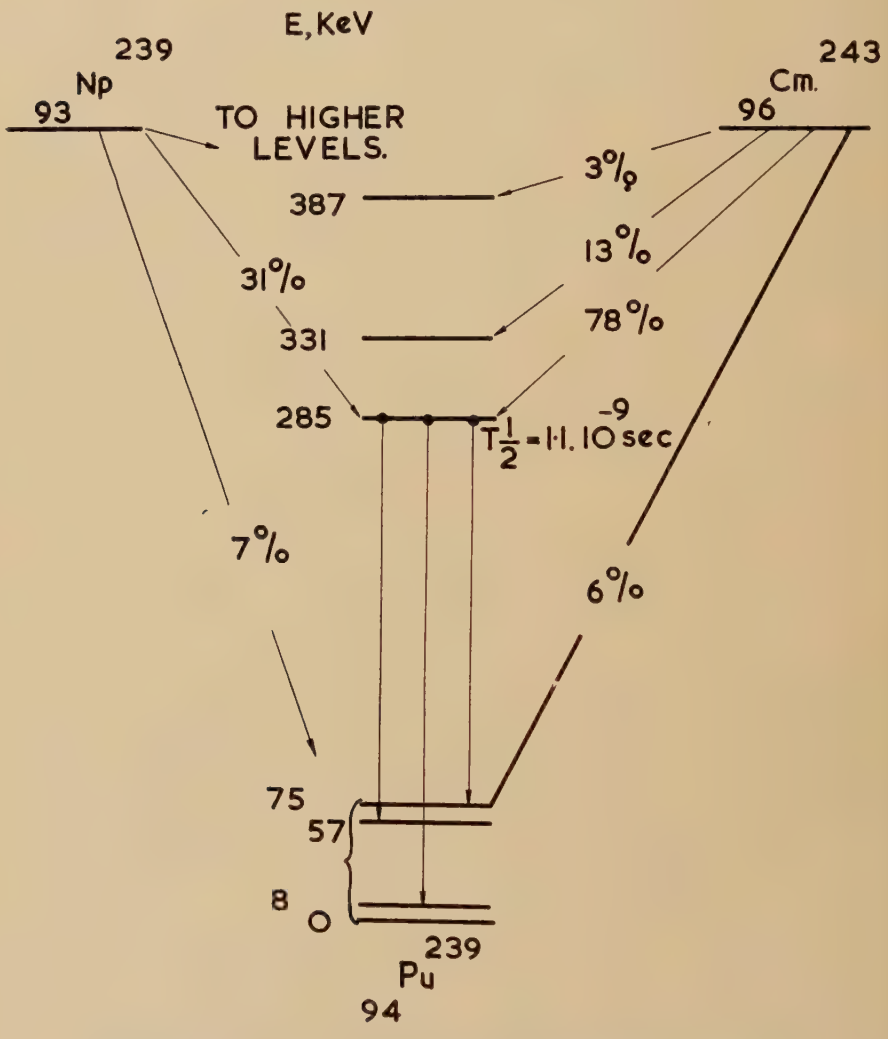
Nucleus	E_2/E_1	$E_3\ E_1$
^{234}U	3·295 ± ·010	6·816 ± ·022
^{238}Pu	3·312 ± ·005	6·891 ± ·011
theoretical	10/3	7·00

The lifetimes for the individual gamma rays are approximately 10^{-8} sec. whilst the Weisskopf single particle formula predicts lifetimes of about 10^{-3} sec and 10^{-1} sec for $E3$ and $M3$, 10^{-9} sec and 10^{-6} sec for $E2$ and $M2$ and 10^{-14} sec and 10^{-12} sec for $E1$ and $M1$ radiations. Transitions which are faster than the single particle predictions by as much as a factor of 10^2 are rarely observed except in the $E2$ case where collective effects are important. It therefore seems safe to conclude that these transitions are dipole or quadrupole.

From our figure for the intensity of the K–X-radiation relative to that of the three gamma rays and with the assumption that the fluorescent yield of the K-shell in plutonium is 0·94 Burhop (1952) we obtain an average conversion coefficient of 2·48 ± 0·25 for the three transitions. According to Slätis (1948) and Fulbright (1949) the K-conversion lines of the three transitions have roughly equal intensities so that the K-conversion coefficients must all be of the order of 3. The $E1$ and $E2$ K-conversion coefficients for gamma rays of this energy are of the order of 0·1 (Rose *et al.* 1951) hence it follows that the observed transitions must be predominantly magnetic in character. If the transitions are all pure

$M1$ the average conversion coefficient should be $3{\cdot}2\pm0{\cdot}2$ whilst if they are all pure $M2$ it should be $7{\cdot}2\pm0{\cdot}5$. The errors on these quantities are difficult to estimate because the conversion coefficients for the 210 and 228 kev transitions have to be derived by extrapolation from the calculated results of Rose *et al.* (1951).

Fig. 10



The decay schemes of ^{243}Cm and ^{239}Np and energy levels in ^{239}Pu.

In order to obtain agreement with the experimental average conversion coefficient, we have to assume that on the average the transitions are either (*a*) $0{\cdot}75\pm0{\cdot}25$ $E2$ or (*b*) $0{\cdot}33$ $M2+0{\cdot}67$ $E1$. Of these (*a*) is more likely since (*b*) would require $M2$ transitions about 30 times faster than

the single particle estimates. We may obtain further evidence on this by estimating K/L ratios for the three transitions from the curve given by Slätis (1948) and comparing them with the theoretical K/L ratios. This comparison is given in table 5 ; the L-conversion coefficients were obtained from calculated data privately circulated by M. E. Rose. The $M1$–$E2$

Table 5. Comparison of experimental (Slätis, 1948) and calculated K/L ratios. The errors are our estimates

$E\gamma$ kev	209	226	277
K/L (expt)	4·4±0·7	4·7±0·9	6·5±1·5
K/L 0·75 $M1$ ±0·25 $E2$	5·0	5·4	5·4
K/L 0·67 $E1$ ±0·33 $M2$	2·4	2·4	2·9

mixture is clearly favoured by this evidence. More detailed measurements on the conversion spectrum are required to obtain the individual mixture ratios for each transition.

It follows from the above result that the average $M1$ lifetime is about 10^{-8} sec and that the average $E2$ life is about 4×10^{-8} sec. The $M1$ transitions are therefore highly retarded by a factor of 2×10^{4}, whilst the $E2$ transitions are retarded by a factor of 20, relative to the single particle estimates.

It seems likely from the results reported by Asaro and Perlman (1954) that the states at 285, 331 and 387 kev in ^{239}Pu (see fig. 10) belong to a rotational band having $K=5/2$ or $7/2$; the four low-lying states probably belong to a rotational band having $K=\frac{1}{2}$ (Newton 1955). Magnetic dipole radiation between these bands will therefore be K–forbidden (Alaga *et al.* 1955). This fact may possibly account for the large retardation factor of the $M1$ radiation. The $E2$ radiation will be allowed if the upper band has $K=5/2$ and K–forbidden if it has $K=7/2$. The smallness of the retardation factor and the scarcity of data on K-forbidden transitions makes it unsafe to draw any conclusion from this result at present.

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CIII. *The Elastic Scattering of 136 MeV Neutrons by Nuclei*

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Abstract

The neutron elastic scattering cross sections for lead, cadmium, copper, aluminium, oxygen, nitrogen, carbon, beryllium and lithium have been measured for an effective neutron energy of 136 Mev. A large liquid scintillator was used, and the cross sections were obtained on an absolute scale.

The integrated total elastic cross sections are consistent with previous measurements of total and inelastic cross sections.

§ 1. Introduction

High energy neutrons are expected to show diffraction patterns in small angle scattering off nuclei. This has been studied at 290 Mev by Ball (1952), 84 Mev by Bratenahl *et al.* (1950), and at several lower energies. The width of the central diffraction maximum depends upon the nuclear radius. Consequently a study of these maxima enables determinations of nuclear radii to be made. The earlier measurements have not been made at angles small enough to give a reliable extrapolation to 0°. On the assumption of an optical model, the value of the scattering cross section at very small angles can be used to yield a value for the volume integral of the nuclear potential which is largely independent of other parameters, as discussed in an accompanying paper (Wilson 1956).

§ 2. Method

The same method was employed as in the n–p scattering experiment (Thresher *et al.* 1955 a) . The absolute differential scattering cross section was obtained by measuring the ratio of the scattered intensity to the direct beam intensity.

With the counter at an angle to the direct beam the maximum available neutron intensity was employed, while the beam was reduced by a factor of about 300 in order to obtain direct beam normalization. When the full beam was employed, the monitor counting rate no longer varied linearly with neutron beam intensity. This effect, together with a possible

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change in proton energy spectrum with intensity, were not separated and a correction for both was determined experimentally. This correction was $16\pm1\%$ for the two effects combined, and compares with $14\pm1\%$ found in the n–p experiment. We allow 2% uncertainty in this.

§ 3. Scatterers

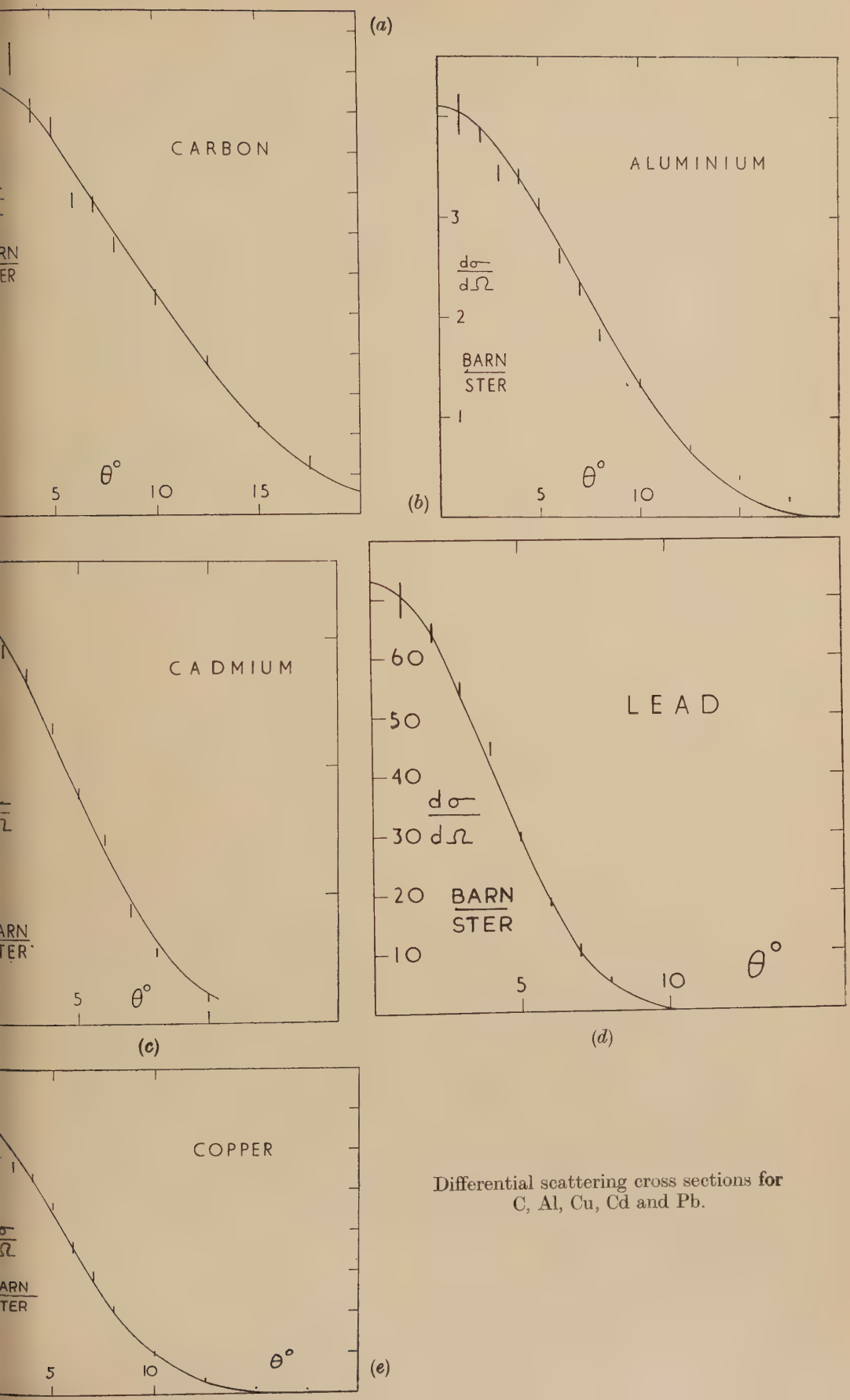
Lead, cadmium, copper, aluminium, oxygen, nitrogen, carbon, beryllium and lithium scatterers were used. They were chosen for the following reasons : (1) the values of A are fairly evenly spaced through the periodic table, (2) a comparison could be made directly with previously determined neutron cross sections (Taylor and Wood 1953, Culler 1955), elastic scattering cross sections (Bratenahl *et al.* 1950, Ball 1952), and inelastic scattering cross sections (Voss and Wilson 1956), (3) the elements were readily available in adequate purity.

The area of the neutron beam was 7·6 cm × 2·54 cm, so that the areas of the scatterers were chosen conveniently larger than this. The thicknesses of the scatterers were chosen to strike a compromise between a conveniently high counting rate and a reasonably small multiple scattering correction. The lead, cadmium, copper, aluminium and lithium were cast into rectangular forms, while beryllium metal powder was obtained already pressed into this form. Liquid nitrogen and oxygen were held in the liquid target container as in the n–p scattering experiment (Thresher *et al.* 1955 a). The lead, cadmium, copper, and aluminium targets were chemically pure to better than 99·99%. These samples were also without flaws. Stereoscopic x-ray photographs were taken of the beryllium and lithium to test for internal flaws and impurities. The beryllium had several flaws. Iron filing impurities were seen with the x-rays and their amount was estimated as $\frac{1}{2}\%$ by volume. A further $\frac{1}{2}\%$ oxygen impurity was also estimated. The density of the lithium was 3% high compared with the expected value, while x-ray analysis indicated about 5% impurity, probably lithium oxide from oxidization in casting by volume.

The nitrogen and oxygen scatterers were moved into and out of the beam manually, while the remaining scatterers were clamped into position on a remote controlled target changer. These scatterers, together with a blank, could be interchanged in any order. This enabled sets of angular runs and the corresponding backgrounds to be determined with a minimum time lapse between runs.

The counter and electronics were the same as in the n–p scattering experiment. Some considerable time had elapsed since the n–p scattering measurements were made (Thresher *et al.* 1955 a) and, moreover, the beryllium target for producing the neutron beam for the present experiment was placed at a slightly smaller radius in the cyclotron than before. Consequently the effective energy of the neutron beam, the effective area of the counter, the linearity factor of the counter and monitor, and the pile-up correction were all redetermined.

Fig 1



Differential scattering cross sections for C, Al, Cu, Cd and Pb.

§ 4. Effective Area of the Counter

The geometrical cross section of the counter was $14 \times 7{\cdot}6$ cm. Owing to edge effects, the effective area of the counter was less than the geometrical cross section, and this had to be determined for the absolute cross section measurements.

A uniform neutron flux was made available at a distance from the counter The neutrons which passed through a precision $7{\cdot}6 \times 2{\cdot}54$ cm steel collimator in a steel well were counted by the counter and then neutrons counted without the collimator. The ratio of counting rates gave the ratio of the effective area of the counter to the $7{\cdot}6 \times 2{\cdot}54$ cm area. The effective area in the present experiment was measured as $72{\cdot}3 \pm 3{\cdot}0$ cm^2 as compared with $73{\cdot}0 \pm 1{\cdot}3$ cm^2 at a slightly higher effective energy in the n–p scattering experiment (Thresher *et al.* 1955 a).

The effective area measurement only applied when the area of the neutron beam striking the counter at the 0° normalization position was exactly $7{\cdot}6 \times 2{\cdot}54$ cm^2. This was very nearly the case for the 6 ft position. At 12 ft however, the area of the beam was 25% larger, and the counts in the direct beam were 8% lower. Accordingly the 12 ft points were normalized to the 6 ft points. Further, the edges of the beam were not sharply defined at the 6 ft position, so that there was an uncertainty in the application of the effective area at this position. A correction and error of (3 ± 2)% was made on the direct beam normalization to allow for this

§ 5. Pile-Up

Pile-up phenomena have been fully discussed in an earlier paper (Thresher *et al.* 1955 b). Pile-up corrections of counts were only significant for normalization runs and for scattering off the heavier elements at smaller angles. In no single count did this correction exceed 3% in the present experiment.

§ 6. Effective Energy

The spread of neutron energies accepted in this experiment was about 20% full width at half maximum. Had the biases been set at higher values in order to reduce the energy spread, the counting rate would have been reduced and become extremely dependent on electronic stability. The ' effective energy ' was determined by measuring the total attenuation cross section of neutrons by carbon in good geometry (Thresher *et al.* 1955 a, b) since this is well known as a function of neutron energy (Taylor and Wood 1953, Culler and Waniek 1955). A value of 136 ± 4 Mev was obtained by this method, assuming that the cross section varies little with energy.

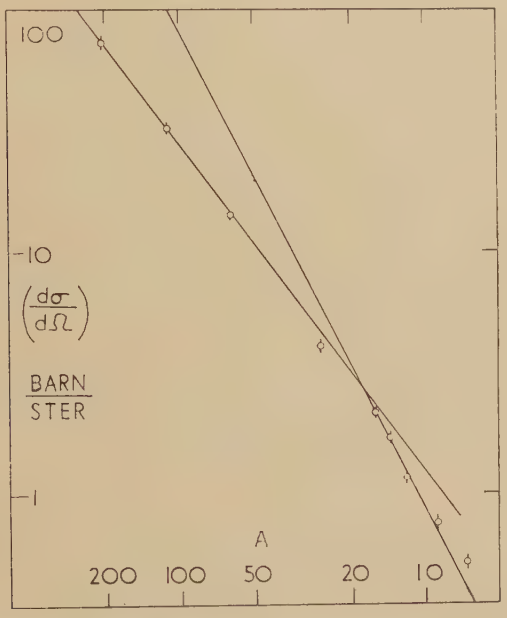
At angles near the diffraction minimum the cross section can vary as fast as $1/E^3$ and an effective energy as low as 129 Mev was possible. For this and other reasons little accuracy was attached to the points near the diffraction minimum.

The error quoted in the effective energy is ± 4 MeV. Of this, ± 3 MeV was due to the uncertainty in Taylor and Wood's energy scale, while the rest was due to statistical errors in the effective energy determination.

§ 7. Background Correction

The neutron background for the counter set at an angle to the neutron beam was determined in a separate run for each angle. At small angles the background decreased rapidly with increasing angle, but at larger angles was nearly constant.

Fig. 2



Differential scattering cross sections extrapolated to 0° plotted as a function of A. The lines are $d\sigma/d\Omega$ proportional to $A^{4/3}$ and A^2.

At small angles the background was appreciably affected by the presence of the scatterer. This has been discussed in the n–p scattering paper (Thresher *et al.* 1955 a). The background consisted of two parts. The first was due to neutrons which had filtered through the shielding direct from the cyclotron target. This was approximately constant irrespective of counter position. The second part was due to small angle scattering of the direct beam, partly by the air and partly by the collimators. This was appreciable only at small angles. The first part was not affected by the presence of a scatterer, whereas the second part, due to the arrangement of the shielding, had to pass through the scatterer and consequently was attenuated. This attenuation was calculated from the total cross sections of Taylor and Wood (1953) for each element. The value of the background at the 10° position was assumed to be the constant background, and hence

the corrected background could be calculated. In no case did the attenuation of the background modify the scattering cross section by more than 3%.

The background due to small angle air scattering was reduced by inserting a long polythene bag containing helium in the beam path, in order to displace the air.

Since at large angles the presence of the scatterers did not affect the background appreciably, a straightforward subtraction of the background gave the necessary correction.

The uncertainty in this background subtraction procedure was small, and was included in the errors.

§ 8. Angular Resolution

In order to determine the angular resolution of the counter the following procedure was adopted. The counter was moved across the beam, keeping the axis parallel to the beam, and the counting rate was plotted as a function of position. The resulting curve is a combination of the angular resolution of the counter and the size of the beam. This was both to find the angular resolution and the effective centre of the counter. The full resolution width at half maximum was found to be 2° and 1° when the counter was 6 ft and 12 ft respectively from the counters.

No correction for the angular resolution of the counter was necessary where the rate of change of elastic scattering cross section with angle was constant. However, where this varied rapidly with angle, corrections were necessary. The correction became greater than 3% only for the scattering cross section for lead for angles larger than 7°.

§ 9. Multiple Scattering

It was necessary to apply a correction to the data to account for multiple scattering of the neutrons in the scatterers. The method used to evaluate this correction has been described by Hildebrand (1951). A neutron which is not absorbed makes on an average t/λ_e collisions in passing through a slab of thickness t, where λ_e is the mean free path for neutron elastic scattering for the element. The probability of just n elastic collisions is given by the Poisson distribution

$$\frac{1}{n!}\left(\frac{t}{\lambda_e}\right)^n \exp\left(-\frac{t}{\lambda_e}\right).$$

If this is multiplied by the probability that the neutron is not absorbed in the thickness t then the probability of n elastic collisions, and no other collisions, in passing through the slab is

$$\frac{1}{n!}\left(\frac{t}{\lambda_e}\right)^n \exp\left[-\left(\frac{t}{\lambda_e}+\frac{t}{\lambda_i}\right)\right] = \frac{1}{n!}\left(\frac{t}{\lambda_e}\right)^n \exp\left(-\frac{t}{\lambda_T}\right)$$

where λ_i is the mean free path for inelastic collisions, and λ_T is the mean free path for a scattering event which is either elastic or inelastic. The above expression assumes that only small angle scattering is involved, so that t is unique.

The central maximum of the scattering pattern may be approximated by a Gaussian distribution. Since transverse momenta comprise a ' random walk ' problem, the customary assumption can be made that the angular spread after n scatterings is $\sqrt{n}$ times the spread from a single scattering. This implies that the forward scattering intensity is reduced by a factor $1/n$.

Thus the relative number of neutrons scattered into unit solid angle at an angle θ to the direct beam by a slab of thickness t is given by

$$I(\theta) = \sum_{n=1}^{\infty} \left\{ \left[\frac{1}{n!} \left(\frac{t}{\lambda_e} \right)^n \exp\left(-\frac{t}{\lambda_T} \right) \right] \left[\frac{1}{n\sigma_e} \frac{d\sigma}{d\Omega} \left(\frac{\theta}{\sqrt{n}} \right) \right] \right\}.$$

In calculating this correction it was found that only double scattering contributed appreciably. In the region of central diffraction maxima this correction was about 4%.

§ 10. Small Angle Magnetic Moment—Coulomb Field Scattering

The elastic scattering cross section at small angles includes, in addition to the nuclear scattering, a contribution due to the interaction of the magnetic moment of the neutron with the Coulomb field of the nucleus (Schwinger 1948, Voss and Wilson 1956 b). A calculated value was subtracted from the measured cross section to yield the nuclear scattering.

§ 11. Total Elastic Scattering Cross Section

The total elastic scattering cross section σ_e was obtained by graphical integration of the differential cross section for scattering of neutrons into a forward cone extending to the largest angle at which measurements were made, namely $17\frac{1}{2}°$.

The intensity of neutrons scattered into wide angles (greater than $17\frac{1}{2}°$) by the heavy elements is so low that measurements become very inaccurate. However, because of the sine factor which weights the differential cross section, the contribution due to these neutrons may not be negligible. Consequently, the wide angle scattering was estimated using the differential cross section measurements made with protons (Dickson *et al.* 1955). The effect of the wide angle scattering is more serious for the lighter elements, and hence the measured total elastic cross sections for these elements are less accurate. The measured values of σ_e are given in table 2, together with the contribution due to wide angle scattering, and an estimated error.

The differential scattering cross sections were expected to vary as $1/E$ and the effective energy of the total cross sections became 134 ± 4 MeV.

§ 12. Contributions from Inelastically Scattered Neutrons

The energy spread allowed in this experiment permitted an appreciable number of inelastically scattered neutrons to be counted. Accordingly it was necessary to determine the possible inelastic contribution to the elastic scattering cross sections.

An indication of the magnitude of this contribution for lead was obtained by a comparison of the upper and lower bias determinations of the cross sections. A diffraction minimum as exhibited by lead, would be obscured by inelastic events if the latter were present in appreciable amounts. Hence, assuming that the upper bias recorded elastic events only, a comparison of the values of the cross sections determined by the two biases gave the contribution due to inelastic events. By this method, the inelastic contribution was less than 20% of the effect in the region of the apparent lead minimum. Assuming the inelastic contribution to be isotropic, this formed less than 0·4% of the effect at small angles. As the upper bias might have been recording a small number of inelastic events an error of 1% was allowed for inelastic scattering.

For carbon a larger error might be present. There are two ways of arriving at an estimate of the error involved. Experimentally the ratio of upper to lower bias determinations could be compared for lead, where the effect is small, and for carbon. This shows that $5\pm5\%$ of the counts at small angles from carbon came from inelastically scattered neutrons. Theoretically, the data on inelastic proton scattering for carbon (Strauch and Titus 1955) can be extrapolated to zero and assumed to apply to neutron scattering. The relative efficiency of counting the inelastic neutrons can be estimated from the bias curve. From this calculation $3\pm2\%$ of the small angle counts are due to inelastically scattered neutrons and about $10\pm5\%$ at 15°. No correction has been made for this error, which is expected to vary as $1/A$.

Note added in proof.—Recent data from Upssala suggest that at small angles the contribution of inelastic scattering is sensibly zero.

Table 1

Elements	C	Al	Cu	Cd	Pb
σ_e up to 17 1/2°	123	311	691	1176	1911
σ_e beyond 17 1/2°	36 ± 10	37 ± 10	52 ± 14	69 ± 17	117 ± 30
σ_c	159 ± 10	348 ± 10	743 ± 14	1247 ± 17	2028 ± 30
$\sigma_t-\sigma_{\text{inel}}$	144 ± 11	334 ± 17	788 ± 27	1336 ± 30	2050 ± 50
Discrepancy (%)	10 ± 11	4 ± 6	$-6\pm3{\cdot}5$	-7 ± 3	-1 ± 3

Weighted average discrepancy $-3\pm2\%$.
Units of 10^{-27} cm^2.

It can be seen from table 1 that for light elements there is a discrepancy between the integrated elastic cross sections and the value of $\sigma_{\text{total}}-\sigma_{\text{inelastic}}$. This discrepancy is within the quoted error, but might disappear if the correction is made for inelastically scattered neutrons.

§ 13. Errors

Two types of errors may be distinguished.

Firstly, relative errors on each point. In every case, except lithium and beryllium, the relative errors quoted on all points were statistical. Due to impurities, an additional error of 4% and 8% was attached to the

small angle lithium and beryllium points respectively. In addition there is the error due to inelastically scattered neutrons discussed in § 12.

Secondly, errors on the absolute scale. These absolute errors were determined by two independent methods :

(i) the direct beam normalization procedure, in which the error was 6% and included the uncertainty in the effective area, linearity factor of the counter and monitor and contributions from inelastic scattering;

(ii) a comparison of the total elastic scattering cross sections with the value of $\sigma_T - \sigma_{in}$ from the measurements of Taylor (1953) and Voss (1956a). This comparison is shown in table 2 and yields a value accurate to 4%. A discrepancy of 3% exists between the total cross sections determined by the two methods.

The direct average of methods (i) and (ii) has been taken and an error of 4% is assigned to the absolute scale.

Only relative errors have been drawn on the cross sections for the angular distributions (fig. 1 (*a* to *e*)). The angular uncertainties of these distributions were due to the finite thickness of each scatterer. The relative and absolute errors were combined, however, to give the quoted errors on the total cross sections and forward scattering cross sections. Theoretical

Table 2. Differential Cross Sections in units of 10^{-23} cm^2; an additional 4% absolute error should be assigned

θ	C	Al	Cu	Cd	Pb
0°	1·190±0·06	4·15±0·2	14·60±0·5	33·00±1·4	73·10±3·2
1°·3′	1·180±0·05	4·08±0·2	14·30±0·4	31·40±1·2	71·10±2·8
2° 6′	1·140±0·04	3·87±0·1	13·50±0·4	29·50±0·5	65·40±1·3
3°	1·150±0·04	3·50±0·08	11·30±0·3	27·60±0·5	55·90±1·2
4°	1·010±0·03	3·47±0·07	10·90±0·2	23·40±0·4	45·30±0·9
5°	0·970±0·03	3·19±0·06	9·40±0·2	18·30±0·4	30·20±0·6
6°	0·790±0·02	2·66±0·08	7·38±0·25	14·50±0·4	18·70±0·5
7°	0·780±0·02	2·31±0·08	5·97±0·25	9·00±0·4	11·30±0·4
8°	0·680±0·02	1·86±0·06	4·26±0·20	5·70±0·22	5·70±0·3
10°	0·550±0·02	1·37±0·04	2·08±0·08	2·11±0·06	1·20±0·06
11° 15′					0·70±0·13
12° 30′	0·390±0·02	0·71±0·02	0·70±0·03	0·52±0·04	0·97±0·06
13° 45′					0·93±0·06
15°	0·226±0·007	0·39±0·02	0·27±0·03	0·33±0·03	0·99±0·07
17° 39′	0·138±0·009	0·17±0·02	0·19±0·03	0·40±0·03	0·53±0·06

θ	Li	Be	N_2	O_2
0°	0·520±0·03	0·760±0·05	1·720±0·08	2·190±0·09
2° 12′	0·510±0·02	0·730±0·05	1·630±0·05	2·070±0·06
4°	0·500±0·02	0·700±0·04	1·480±0·05	1·800±0·05
15°	0·140±0·010	0·194±0·010	0·270±0·02	0·288±0·009
17° 39′	0·095±0·010	0·113±0·007	0·138±0·007	0·172±0·009

curves for a square potential well are fitted to the data with arbitrary normalization on both axes. Three curves fit well, but the carbon and cadmium curves if fitted at zero degrees, fall off the experimental points. This may indicate a systematic error in these elements.

§ 14. Extrapolation to Zero.

The points were extrapolated to zero by using the 1° and 2° points and assuming that the cross section varies as $1-\alpha\theta^2$ in this region (except for cadmium where a fit to the whole curve was used).

The nitrogen, oxygen, beryllium and lithium data were extrapolated to zero assuming corrections intermediate to those for carbon and oxygen. The results are included in table 2.

A transition from the transparent nucleus formula (A^2) to the opaque nucleus ($A^{4/3}$) is seen on proceeding from light to heavy nuclei.

§ 15. N–P Differential Cross Section

In the work of Thresher *et al.* (1955 a) preliminary results of this experiment threw doubt upon the normalization and a large error was assigned to the n–p scattering. The discrepancies were found in the effective area measurement and are now removed, and we believe that the differential cross section scale for the 137 MeV n–p scattering is accurate to 5% if the points are *raised* by 2%. It is possible that the similar, but smaller discrepancy at the 105 MeV points is due to the same cause. We cannot be sure about this, so we believe that the quoted uncertainty of 6% should be retained.

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CIV. *The Nuclear Radius and Potential from Neutron Diffraction Scattering*

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Abstract

Neutron diffraction scattering at high energies is discussed and it is shown that two parameters can be derived from the data; the nuclear radius and the nuclear potential, both with little effect of other parameters. These parameters are derived for each nucleus individually. The radius, so derived, is larger than the charge distribution radius. The potential agrees with that derived from the analysis of total cross sections of all elements simultaneously when a nuclear potential with a reasonable taper is assumed.

§ 1. Introduction

Neutron diffraction scattering measurements have been made at 84 Mev, 136 Mev and at 290 Mev and at several lower energies (Van Zyl *et al.* 1956 preceding paper, and references therein). The purpose of this paper is to discuss their interpretation on the optical model of the nucleus, using the Huyghen's principle approximation for computation (Fernbach *et al.* 1949). In particular two experimental features will be discussed, the value of the differential cross section extrapolated to zero angle, and the width of the diffraction pattern.

In this paper a departure from the usual course will be made. Each nucleus will be considered independently upon the optical model, and only later will the parameters of two nuclei be compared. This is possible for this paper, where more than one measurement is considered for each nucleus. It was impossible for previous optical model comparisons (e.g. Taylor 1954) which considered only total cross sections. In this respect, the work may be regarded as a continuation of that of Voss and Wilson (1956).

Fernbach's original equations can be generalized to include effects of a diffuse nuclear surface, and a spin–orbit coupling. This generalization has been discussed before (Malenka 1954, Wilson 1955, Voss and Wilson 1956). The notation used here will be the same as that used in the last

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two references. The scattering amplitude for scattering from a potential $(1+i\epsilon)V(r)+\vec{S}.\vec{L}U(r)$ may be written in the form

$$f(\theta)+\vec{\sigma}.\vec{n}g(\theta)$$

where
$$f(\theta)=ik\int_0^\infty pdp\ \{1-\exp[-2i\delta_0(\rho)]\cos[kp\delta_1(p)]\}J_0(kp\theta) \quad (1)$$

$$g(\theta)=ik\int_0^\infty pdp\ \{\exp[-2i\delta_0(\rho)]\sin[kp\delta_1(p)]\}J_1(kp\theta) \quad . \quad (2)$$

where
$$\delta_0(\rho)=-\frac{k}{2E}(1+i\epsilon)\int_0^\infty V(r)dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\delta_1(\rho)=-\frac{k}{2E}\int_0^\infty U(r)dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and
$$r^2=\rho^2+z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

k is the wave number of a neutron of energy E.

In the analysis which follows the contribution to $U(r)$ from the interaction of the magnetic moment of the neutron and a coulomb field will be ignored, and the appropriate correction made to the experimental results, so that, when interpreted, they will yield the nuclear potential without the magnetic moment interaction. A correction of about 1% at the small angles has already been subtracted from the experimental points in the work of Van Zyl.

$U(r)$ may now be assumed to have a reasonable form which falls to zero outside the nucleus. ($U(r)=0$ for $r>R$). Then $g(\theta)$ varies as θ for small θ and if the experimental neutron scattering cross sections are extrapolated to zero angle, the value determined, which will be called I (0), is determined theoretically by $f(0)$ alone. The angles over which the measurements are made are small enough that $2\sin\frac{1}{2}\theta=\theta$ within experimental error, so that Huyghen's principle should be valid.

§ 2. Validity

The approximation is only expected to be valid, even for small θ, if V/E is sufficiently small. Shapiro (1955) has compared the angular distribution of neutron diffraction sattering using both the small angle approximation and an exact phase shift analysis. He finds that $I(0)$ is larger for the exact calculations, and the value of $I(\theta)/I(0)$ is reduced by about 10% in the region of angles of interest here. This alters the value of R determined by the width of the diffraction pattern by $1\frac{1}{2}$%. For Shapiro's calculations V/E was of the order of 0·25 whereas V/E here is approximately 0·12. Moreover the important calculations of this paper assume a tapered potential whereas Shapiro used a square well potential so that neglect of reflection at the nuclear boundary is justified. Accordingly the correction due to the use of the Huyghen's principle approximation is less than 1% and is neglected. It is of interest to note that the exact

calculations quoted give a value for σ_T about 10% greater than the optical model. If σ_T were used to derive a value of R, there would be a 2% error instead of the $\frac{1}{2}$% here.

In the determination of V from small angle scattering, the experimental errors are also sufficiently large to mask any errors.

The neutron diffraction cross sections are determined experimentally in the frame of reference where the nucleus is at rest ; and so are the total neutron cross sections. Thus $Re\ f(0)$ is determined in this frame. It would be possible to transform $Re\ f(0)$ into the centre of mass system. The energies would now be different for the different nuclei, and would fall from 136 Mev to 120 Mev for carbon. The question naturally arises, in what frame of reference are the optical model parameters expected to be most nearly constant from nucleus to nucleus ? So long as attention is confined to the parameters R, k_1 and $\overline{K}$, where $k_1+i\frac{1}{2}K$ is the change in wave number on entering the nucleus, then there is no difference so long as Huyghen's principle is valid. The situation is different when it is desired to derive a value for the potential V. This question can be answered by considering the derivation of the nuclear potential from individual nucleon–nucleon interactions (Jastrow 1951, Mandl and Skyrme 1952.) The same problem arises in the interpretation of nucleon–deuteron scattering (Chamberlain and Stern 1954). The nuclear potential may be derived from the formula

$$V=\frac{3E}{k^2R^3}\{Z[\tfrac{3}{4}fnp^t(0)+\tfrac{1}{4}fnp^s(0)]+(A-Z)[\tfrac{3}{4}fnn^t(0)+\tfrac{1}{4}fnn^s(0)] \quad . \quad (6)$$

which is derived by expressing the forward scattering amplitude for neutron–nucleus scattering as the sum of singlet and triplet neutron–proton (fnp^s](0)) and $fnp^t(0)$ and neutron–neutron ($fnn^s(0)$ and $fnn^t(0)$) scattering amplitudes (with the spin-flip terms removed). This form is valid in the impulse approximation and from its structure clearly applies to the frame of reference in which the nucleus is at rest. Thus the appropriate frame of reference to analyse the zero angle cross section is expected to be that in which the target nucleus is stationary. This is the procedure that has been adopted in the past. This argument must be used with caution, however. The same approximation would show that the nuclear potential has the same radius as the matter distribution which will be shown to be incorrect.

It remains to be shown that the use of a potential to describe the interaction of the neutron with a nucleus is sufficiently accurate. No exact analysis is attempted here, but the problem is related to that of x-ray scattering (e.g. Compton and Allison 1935). Here Born approximation is valid for the scattering from an atom, and the form factor calculated is the same as would be calculated upon the optical model using a potential. But in this case, the form factor must be multiplied by the cross section for scattering from an individual electron, which is assumed isotropic in the optical model.

By analogy it would appear that the calculations using the optical model must be multiplied by the square of the scattering amplitude

$$Z\,[\tfrac{3}{4}fnp^t(\theta)+\tfrac{1}{4}fnp^s(\theta)]+(A-Z)\,[\tfrac{3}{4}fnn^t(\theta)+\tfrac{1}{4}fnn^s(\theta)] \quad . \quad . \quad (7)$$

The $fnn(\theta)$ are probably nearly constant with θ near 0°, but the $fnp(\theta)$ vary with θ (Thresher *et al.* 1955). For light elements we are concerned with fnp to 15°, where $Inp\ (\theta)\simeq 0{\cdot}7\ Inp(0)$. For lead we are only concerned with fnp to 6° where $Inp\ (\theta)\simeq 0{\cdot}95\ Inp(0)$.

The internal momentum of the nucleus in the nucleus will reduce the effect somewhat since it spreads out the angles involved. Thus an internal momentum of 20 Mev will spread the angles over 21°. Experimentally, Chamberlain and Segre (1952) has found an angular spread of 45° (at half maximum) for quasi p–p collisions in Lithium nuclei, using 340 Mev bombarding energy. At 135 Mev this spread should be increased to 70° which reduces the effect to insignificance.

If this is regarded as a correction to the calculations of this paper and the momentum distribution is ignored, then the radius determined for lead must be multiplied by 0·985 and that for Lithium by 0·86. Due to the uncertainties involved, no correction has been assigned to the data, but this consideration should be considered as giving the order of accuracy to be expected from the calculations.

§ 3. Width of Diffraction Pattern

For a nucleus which is nearly opaque to the incident neutrons, the diffraction pattern has the angular distribution

$$I(\theta)=\left|\frac{J_1(kR\theta)}{kR}\right|^2 k^2R^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Transparency of the nucleus broadens the diffraction pattern slightly and reduces the diffraction minima associated with eqn. (8). But in the limit of a completely transparent nucleus (Born approximation) the value of θ for $I(0)/I(\theta)=4$ is only increased by 10% from that for an opaque nucleus. The diffuseness of the nuclear boundary also has an important effect upon the diffraction pattern, filling out the minimum.

Here, we choose the angle at which $I(0)/I(\theta)=4$ and endeavour to extract from this value information about the nuclear radius. This choice is influenced by the fact that at such an angle the plot of $I(\theta)$ versus θ is steep, so that he accuracy of determination is quite large. Also, it is the hope that at this angle the experimental errors arising from the detection of inelastically scattered neutrons (discussed at some length by Van Zyl 1956) are small. It has only been thought worthwhile to carry through this analysis for 136 Mev where the best data are available. The other data scale to the 136 Mev data by the relation $\theta^2/E=$constant but the errors are considerable.

When the effect of the diffuseness of the boundary is taken into consideration, it is found using Born approximation that the value deduced of

$$\langle r^2\rangle=\int V(r)r^2d\tau\Big/\int V(r)d\tau \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is not appreciably affected by the degree of taper used. This result has already been found for electron scattering.

For reasonable optical model parameters the difference is not large and can be readily computed. Therefore the most accurate radius parameter deduced from our data is the mean square radius. The detailed calculations support this conclusion. The angular distributions of Van Zyl are in agreement with the Born Approximation shape, as may be seen from the figures of the previous paper, but addition of a taper slightly improves the agreement.

§ 4. Spin–Orbit Coupling

A correction is applied for the presence of spin–orbit coupling in the nucleon–nucleus interaction which broadens the angular distribution. This correction is semi-empirical and is derived as follows.

The unpolarized cross section is given by

$$I(\theta)=\{f(\theta)\}^2+\{g(\theta)\}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and the polarization in the scattering is given by

$$P(\theta)=\frac{2\{Imf(\theta)Img(\theta)+Ref(\theta)Reg(\theta)\}}{I(\theta)} \quad . \quad . \quad . \quad . \quad (11)$$

For light nuclei within the central diffraction maximum,

$$Imf(\theta)/Re\,f(\theta)\simeq Imf(0)/Re\,f(0)=A \quad . \quad . \quad . \quad . \quad (12)$$

where A is close to unity. $g(\theta)$ is mainly imaginary but its real component is of such a sign as to reduce the polarization (e.g. Wilson 1955). Then an approximate minimum value of $g(\theta)$ becomes

$$|g/f|=A[1-(1-P^2-P^2/A^2)^{1/2}]/(A^2+1)^{1/2}. \quad . \quad . \quad . \quad (13)$$

The values of $P(\theta)$ for the various elements are taken from the data of Dickson *et al.* (1955). These data are for protons and it is assumed that the plot of $P(\theta)$ is the same for neutrons and protons except for the coulomb interference region. Here the data are extrapolated assuming $P(\theta)$ is proportional to θ (Wilson 1955). Thus for carbon at 14°, which is the relevant angle, we find $|g/f|=0{\cdot}14$ so that the correction to $I(\theta)$ is 2%. If appreciable phase shift occurs in $g(\theta)$ (which will only occur if a spin–orbit potential is assumed which is *not* the derivative of the nuclear potential) then the correction might be up to 5%. $3\frac{1}{2}\%\pm1\frac{1}{2}\%$ has been taken as the correction. For heavy nuclei there is certainly some phase shift occurring ; the correction on the above basis at 6° becomes 0·02% ; if phase shift occurs the correction could be 1% which is still negligible.

There remains the effect that the spin orbit coupling has on $f(\theta)$. The effect here is to *reduce* the contribution that large p make to $Re\,f(\theta)$, and to increase the contribution of large p to $Imf(\theta)$, making the nucleus more nearly opaque.† The effect on a determination of the nuclear radius is, therefore, small.

† A discussion of the effect on $Imf(0)$ and therefore σ_T in Wilson (1955) contains an error which did not, however, alter the arguments of that paper. The discussion here is correct.

In order to make an exact estimate of the effect of the spin–orbit coupling a magnitude and share of the potential $U(r)$ must be assumed. There is by now considerable evidence that $U(r)$ must be nearly proportional to $1/r\{dV(r)/dr\}$ (Wilson 1955, Sternheimer 1955), although an imaginary component of $U(r)$ has been assumed by Heckrotte (1956). Accordingly a value of $U(r)$ which gives a good fit to the data on polarization at small angles was assumed. It is not possible to do this consistently for a square well potential $V(r)$ since with such a potential it is not possible to fit the polarization. So that the same $U(r)$ was used for the square well as fits the data using a *tapered* potential.

§ 5. Tapered Potentials

Since a square potential well is unrealistic, the calculations were performed for tapered potential wells. The value of R determined by the data has been calculated using tapered potential distributions with parameters for the potential and absorption coefficient that fit the absorption cross section data, the small angle cross sections, total cross sections, and the polarization at small angles.

For lead the potential is assumed constant to a value $0{\cdot}8R_M$, tapering linearly to $1{\cdot}2R_M$. For copper potential assumed is constant to $0{\cdot}71R_M$ tapering to $1{\cdot}29R_M$. The total cross sections have been calculated for these potentials for a range of parameters (Voss and Wilson 1956), though it should be noted that the notation here is slightly different. For carbon where the details of the taper become important, a Woods and Saxon (1954) potential is assumed with a shape

$$V(r)=\frac{V_0}{1+\exp\ (r-R_M)/a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where $a=0{\cdot}39$ fermis.

All these potentials have the common feature that, for the appropriate nucleus, the distance for the potential to fall from 90% to 10% of its central value is 2·5 fermis in agreement with the taper of the charge density distribution (Hofstadter 1956).

The taper for lead is *less* than that assumed by Williams (1955) at high energies, but seems the most that can be allowed by total cross section measurements (Voss and Wilson 1956).

For each case the value of $\langle r^2\rangle$ has been calculated and the radius *Req* of a square potential well that will yield the same $\langle r^2\rangle$. This *Req* should be nearly as accurate as the experimental data.

The value R_M is dependent upon the taper. A variation of the taper distance by 0·1 fermi gives less that 1/2% change in R_M for heavy nuclei and about 1% for light nuclei. No significant error is expected from the choice of the other parameters which come from experiment, except the polarization which can give 1% uncertainty for light elements. An overall uncertainty of 5% is therefore applied to R_M for the light elements and about 3% to the heavy elements.

It was not necessary to carry through detailed calculations for all nuclei since the corrections are small. The corrections were computed for C, Cu and Pb and applied to the other nuclei by graphical interpolation.

§ 6. Data and Comparison

Table 1 shows the experimental data of Van Zyl and the calculated nuclear radii. The four columns give the values according to the following assumptions. (1) Born approximation (shown as R_{BA}), (2) Opaque nucleus (R_{ON}) (3) A square well potential with Taylor's parameters for V and K and the polarization fitted by a tapering nucleus (shown as R_{SW}), (4) and (5) the ' best ' fit being that calculated from the nuclear potentials just

Table 1. Angle for which $I(0)/I(\theta)=4$ for 136 Mev and values of R(in fermis) ($k=2{\cdot}654\times10^{13}$ cm^{-1})

	Li	Be	C	N	O	Al	Cu	Cd	Pb
Angle (degrees)	15·5 ±0·25	14·9 ±0·2	13·9 ±0·2			11·1 ±0·2	8·45 ±0·15	7·41 ±0·12	6·08 ±0·16
R (B.A.)	3·48	3·62	3·88	4·04	4·20	4·86	6·39	7·28	8·89
R (O.N.)	3·07	3·20	3·43	3·58	3·72	4·29	5·63	6·42	7·83
R (S.W.)	3·41	3·55	3·76	3·96	4·11	4·77	6·13	7·01	8·52
R_M	2·65 ±0·2	2·80 ±0·2	3·06 ±0·15	3·22 ±0·2	3·46 ±0·2	4·08 ±0·15	5·67 ±0·2	6·56 ±0·25	8·00 ±0·2
R_{EQ}	3·37 ±0·1	3·48 ±0·1	3·72 ±0·07	3·83 ±0·1	4·02 ±0·1	4·62 ±0·08	6·03 ±0·1	6·92 ±0·15	8·45 ±0·2

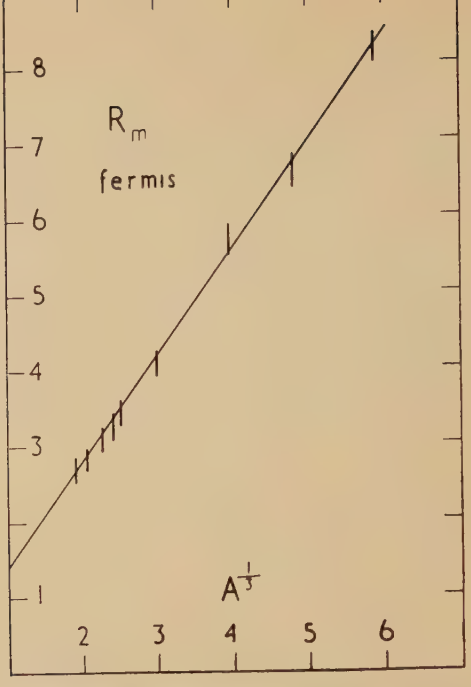
An additional 1½% error is applicable to *all* the values because of the uncertainty in the energy.

described yielding values *Req* (the radius of a square well which would yield the same r.m.s. radius) and R_M (the radius of the point at which the potential falls to one half of its maximum value).

Figures 1 and 2 show plots of R versus $A^{1/3}$ for *Req* and for R_M. The line is the radius of the electron density distribution as determined by Hofstadter.

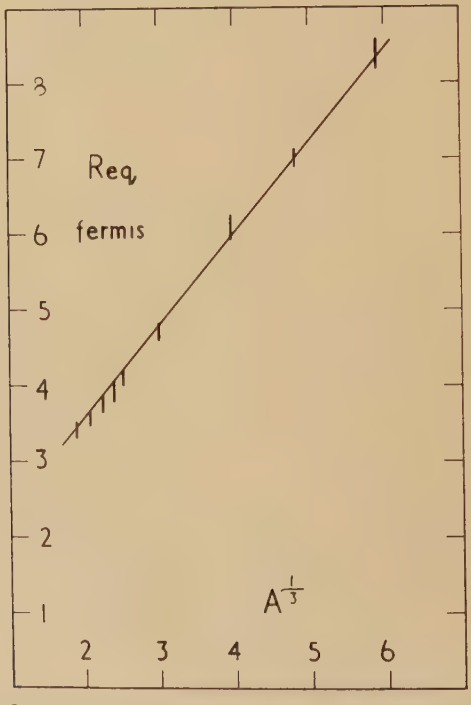
§ 7. Analysis of Zero Zngle Cross Sections

$I(0)$ may be written $|Ref(0)|^2+|Imf(0)|^2$. This suggests at once the use of the cross section theorem to determine $Imf(0)$. This gives $Imf(0)=k\sigma_T/4\pi$. σ_T is known accurately from a variety of experiments over the energy range of interest (Taylor 1952, Culler and Waniek 1955, Nedzel 1954), so that $|Imf(0)|^2$ may be determined to about 2%. The accuracy of the energy measured for σ_T and for $I(0)$ does not enter into



The mean radius as deduced in this paper. The line is $R=1{\cdot}35\,A^{1/3}$. These values depend upon the taper assumed as discussed in the text.

Fig. 2



The 'effective' radius *Req* of a square well that would produce the same scattering. The line is $1{\cdot}2+1{\cdot}18\,A^{1/3}$. All points have an *additional* 1½% uncertainty due to the energy determination.

this determination to any significant extent for the data of Van Zyl, since for that data, the effective energy was determined by a total cross section measurement. A slight additional uncertainty arises, however, in interpreting the data of Bratenahl and Ball.

The use of this cross section theorem is more general than the optical model or any particular computational approximation, since it may be derived entirely from a conservation law. Consideration must, however, be given to the effect of exchange interactions of neutrons with the neutrons in the nuclei, which is neglected upon the optical model, and is tentatively neglected in this paper.

The Huyghen's principle method of calculation is not accurate for very low energies. By this method of subtracting $|Imf(0)|^2$ from $I(0)$ to obtain $|Ref(0)|^2$ we obtain $Ref(0)$ with no uncertainty, and any analysis of $Ref(0)$ becomes more accurate. This is seen particularly in an analysis of the neutron scattering from lead at 84 Mev where $Ref(0)$ is small compared with $Imf(0)$ and could not be determined at all within the uncertainty of an optical model fit to $I(0)$. Here, however, it is possible to determine $Ref(0)$ even in this unfavourable case.

From (2) we can write

$$Ref(0)=k\int_0^\infty pdp \sin[2\delta_0(p)]\cos[kp\delta_1(p)]. \quad . \quad . \quad . \quad (15)$$

This can be expressed in the form

$$Ref(0)=\frac{k^2R^3V}{3E}\{F(V, E, R, U)\}, \quad . \quad . \quad . \quad . \quad (16)$$

where F is a function of V, E, and U, which is less than unity but reaches unity in the limit of the Born approximation. The term $\cos[kp\delta_1(p)]$ acts, for a derivative type spin–orbit coupling, to reduce the integrand for large p where $\delta_0(p)$ already vanishes. Thus even though the spin–orbit coupling and the imaginary parts of the potential are not well known, this measurement can still be used to determine VR^3.

§ 8. Measurements of Zero Angle Cross Sections

In table 2 are the experimental results extrapolated to zero degrees. Also tabulated are the derived values of $|Ref(0)|^2$. No cross section determinations were available at the exact energies required, so that $Imf(0)$ was determined by interpolation between different energies. Cross section determinations were available for the elements Li, Be, N and O at very few energies, so that there is a small error in the interpolation at these energies. This has been estimated and included. The author has arbitrarily assigned errors to the data of Bratenahl and of Ball. The errors at 84 Mev in $|Ref(0)|^2$ range from 30% for Be, C, Al, to about 100% for Cu, Ag and Pb. At 136 Mev errors range from 15% for Li to 30% for Cd and 50% for Pb. At 290 Mev errors are 40% for C and Al and 100% for Cu and Pb. For a given value of Req, a realistic taper gives a different value for $Ref(0)$ than the square well for two reasons.

Table 2. Experimental and derived quantities at zero degrees

$I(0)$ (10^{-24} cm^2/ster)

Element	Li	Be	C	N	O	Al	Cu	Ag	Cd	Pb
84 Mev		1·2	1·7			5·1	15·5	29		63
136 Mev	0·52	0·76	1·19	1·72	2·19	4·15	14·6		33·0	73·1
290 Mev			1·25			5·5	16			130

$|Im\, f(0)|^2$ (10^{-24} cm^2/ster)

Element	Li	Be	C	N	O	Al	Cu	Ag	Cd	Pb
84 Mev		0·508	0·878			3·76	13·7	28·0		58·9
136 Mev	0·227	0·336	0·585	0·768	0·96	2·46	10·22		25·7	62·5
290 Mev			0·827			3·27	14·50			85·5

$|\mathrm{Re}\, f(0)|^2$ (10^{-24} cm^2/ster)

Element	Li	Be	C	N	O	Al	Cu	Ag	Cd	Pb
84 Mev		0·69	0·82			1·3	1·8	1		4
136 Mev	0·29	0·42	0·61	0·95	1·23	1·71	4·4		7·3	10·6
290 Mev			0·42			2·23	1·5			45

Firstly $\int Vr^2 d\tau$ is *smaller* ; secondly the edges of the nucleus are more transparent and consequently the average value of F is more nearly equal to unity. The two effects tend to cancel.

Accordingly $Ref(0)$ has been calculated for the tapered potentials used before using the values of R previously determined and the experimental value has been used to assign a value to V. This is tabulated in table 3. Also tabulated is the value of V that is calculated in Born approximation (showing therefore the effect of nuclear absorption). The results are in satisfactory agreement with a reasonably low value of V as required by Taylor's fit to total cross sections. In table 3 the errors assigned are those of the experiment only ; in addition are errors in R and in the other parameters. These yield an overall error of about 3 Mev in the values of V.

§ 9. Discussion

The values of the nuclear radius and of the potential derived in this paper are derived for each nucleus individually. By contrast, the values

of the radius or potential derived by Taylor or by Shapiro are derived assuming that the potential is the same for all nuclei and that R varies as $a+bA^{1/3}$. It is gratifying that such good agreement is obtained, once an appropriate tapered potential has been chosen. This paper has therefore verified that the optical model parameters are nearly constant for all nuclei. The most accurate radius parameter determined is *Req.* This has been plotted in fig. 1 versus $A^{1/3}$. R_M and *Req* for the charge density distribution have been accurately determined by Hofstadter (1956) and are $R_M=1{\cdot}07\times A^{1/3}$ fermis and $Req.=1{\cdot}18\times A^{1/3}$ fermis for lead and $1{\cdot}35\times A^{1/3}$ fermis for light elements. The difference in the ratio between Hofstadter's results and those assumed here are due to slight differences in the taper assumed.

Table 3. Nuclear Potential (MEV) deduced in this paper

	V_{135} Born Approx.	V_{135}	V_{290}	V_{84}	V_{135} from σ_T
Li	14·2	18·4±1·5			
Be	14·2	19·9±1·3		28±3	
C	13·8	18·1±1·5	14±4	26±3	
N	14·3	19·9±1·3			
O	14·3	20·1±1·4			
Al	10·1	15·4±1·7	15±3	20±4	
Cu	6·6	12·3±1·5	6±6		13·2±1·0
Cd	4·9	11·2±2·3			
Pb	3·6	14 ±5	23±15		14·0±1·0
Taylor		13·5	10	20	

This plot of the nuclear radius shows clearly that the radius of the potential is larger than the radius of the charge distribution. The amount of this difference varies from 0·5 fermis for low atomic numbers to 1·25 fermis for high atomic numbers. The difference for low atomic numbers must be attributed to the effect of nuclear forces (Drell 1955). However if the radius for lithium were multiplied by 0·86, to allow for a variation of V with angle (§ 2), then the 0·5 fermi difference between the potential well and the charge density radii would vanish. The effect of nuclear forces should be about the same for high atomic numbers ; a difference between the radius of the charge distribution and the radius of a matter distribution, if real, would give an additional effect which can be as large as 0·7 fermis.

The plot of nuclear radius versus $A^{1/3}$ seems quite smooth in the region of carbon. The high binding energy of carbon seems to have less than 2% effect on the nuclear radius though there is an appreciable effect in the apparent radius calculated from deuteron stripping cross sections (Holt and Marsham 1953).

§ 10. Comparison with Absorption Cross Sections

The radii obtained at 140 Mev from absorption cross sections are somewhat dependent on the value of α, the value assumed for the effect of the exclusion principle upon nucleon–nucleon scattering inside nuclei. Nevertheless, it is interesting to compare the results of Voss and Wilson (1956) for the nuclear radius, for the heavy elements.

Table 4 compares the nuclear radii *Req* deduced from the angular distributions in this paper, with the values R_N and R_p deduced from neutron and proton absorption cross sections and also R_{charge}—the

Table 4. Comparison of Radius determinations units of 10^{-13} cm (fermis)

	Pb	Cd	Cu
Req	8·5 ±0·2	6·92±0·2	6·03±0·15
R_n	7·65±0·1	6·16±0·1	5·35±0·1
R_p	8·3 ±0·4	7·23±0·45	5·61±0·4
R_{charge}	6·99	5·69	4·68

charge distribution radius. The absorption cross sections only give reasonable values for heavy elements. The same taper has been assumed as in this paper.

It is seen that the values of R deduced fron neutron absorption experiments lie between the value deduced from neutron scattering and electron scattering. This also is *consistent* with the idea that the neutron radius is larger than the proton radius. Alternatively it might suggest that the real part of the nuclear potential has a larger radius than the imaginary part, the latter being fairly close to the charge radius.

Note added in proof.— Recent measurements of the ratio of absorption of 700 Mev positive and negative π-mesons by Cool show that the neutron and proton radii are essentially the same.

It should be noted that although these radii are in agreement with the radii deduced from 84 and 290 Mev neutron data (see § 3), radii deduced from proton scattering depend upon the energy, and at higher energies are smaller than those deduced here. This fact, noted before by Shapiro (1955), may indicate a dependence of radius upon impact parameter.

§ 11. Discussion of the Potential

The value of the nuclear potential derived from this experiment is high compared with that obtained from total cross section measurements. It is however, more definite in the present analysis ; attempts have been made to fit 260 Mev and 400 Mev neutron total cross sections by assuming that $V=0$ at these energies (Nedzel 1954). By this means a linear plot of

R versus $A^{1/3}$ was obtained. However if $V=0$ then $Ref(0)=0$ and it is then impossible to fit the neutron scattering data of Ball.

The discrepancy between the values of V found from this paper and those of Taylor may lie in two of Taylor's assumptions. Firstly the total cross sections for light nuclei are not very sensitive to the nuclear potential, and the value of V was fitted to the heavy nuclei. It is possible that V is slightly different for light and heavy nuclei. Secondly, part of the discrepancy is removed by considering the taper of the nucleus.

There remains the effect of the exchange of particles upon the cross section theorem. In the examples of the breakdown of the theorem (in its elementary form) given by Mapleton (1954) the forward scattering amplitude is *larger* than that given by $k\sigma_T/4\pi$. This would seem to be a general result.

An order of magnitude estimate of its failure can be given by the following indirect argument. The production of neutrons by proton bombardment of nuclei at high energies has been interpreted as a quasi pn scattering with a neutron inside the nucleus. The energy distribution of the outgoing neutrons is in qualitative agreement with a reasonable momentum distribution of the bound neutrons (Mandl and Skyrme 1952). A similar effect should occur for quasi n–n scattering inside the nucleus, and the residual nucleus should be in one of the lower lying levels according to the momentum of the struck nucleon. This inelastic scattering of *protons* has been observed. Some of the neutrons will be elastic, and the extent to which these are exchange neutrons will govern the failure of the cross section theorem.

By a plausible extrapolation of Strauch's data, we find for carbon that inelastically scattered neutrons leaving the nucleus in the 4·4 MeV excited state have an intensity about 4% of the elastic neutrons. We might tentatively estimate the contribution of exchange neutrons to the forward elastic scattering intensity to be about half this or 2%. The effect on the scattering amplitude will then be about 10% with a corresponding effect upon the cross section theorem.

Accordingly, it is possible that the values of V deduced in this paper are high by about 10%—which could eliminate the discrepancy between the results of this paper and those of Taylor.

This conclusion is supported by the calculations of Chew for neutron–deuteron scattering (Chew 1948). Our neglect of exchange corresponds to his most elementary treatment. Inclusion of exchange gives a larger cross section at small angles. The exchange effects in the deuteron also give a *narrower* diffraction pattern, so that the value of the radius, as determined in this paper, may have to be modified.

§ 12. Comparison with Total Cross Sections

It is not certain that the radii and potential shapes of the real and imaginary parts of the potential are the same, and indeed the comparison in table 4 suggests that they are not. In view of this, an extensive

comparison with experiment of total cross sections calculated from the parameters here deduced seems unjustified. For light nuclei also, the shape of the potential is in considerable doubt.

Assuming the inelastic cross sections measured by Voss and Wilson (1956) and the radii here presented and using also the taper assumed, the real part of the nuclear potential is deduced for copper and lead. The results are included in table 3. For carbon the error in this procedure is too great to be meaningful. The total cross section has been calculated from *all* the parameters in this paper and agrees with experiment to 5% within an error of 15%. These comparisons show that there is nothing drastically inconsistent in the procedure.

Acknowledgments

The basic ideas of this paper were developed while the author was still at the Clarendon Laboratory, Oxford. With his colleagues, Dr. R. G. P. Voss and C. P. Van Zyl he has had many discussions. To Professor V. F. Weisskopf lies the credit for the method of analysis of small angle cross sections, though the author is solely responsible for its execution. In conclusion Professors S. D. Drell and R. W. Williams have contributed much valuable criticism.

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CV. *An Investigation of Neutrons from the Reaction* ^{10}B (d, n) ^{11}C *at* 1·08 MeV *Bombarding Energy*

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Summary

Angular distributions and energy spectra of the neutrons from the reaction ^{10}B (d, n)^{11} C have been measured at 1·08 MeV bombarding energy using the photographic plate technique. Energy levels were found in ^{11}C at 2·02±0·04, 4·24±0·03 and 4·73±0·03 MeV. A large part of the reaction cross section must be due to compound nucleus formation. The experimental angular distributions can be adequately represented if a pure stripping curve is added to an isotropic distribution from compound nucleus formation. The orbital angular momentum of the captured protons has been determined.

§ 1. Introduction

Previous measurements have been made in this laboratory (Graue 1955) on the angular distribution of the ground state neutron group from ^{10}B (d, n) ^{11}C. The distribution obtained at 0·85 MeV bombarding energy was found to be almost isotropic, thus indicating a large contribution from compound nucleus formation. Paris and Endt (1954), however, investigated the same reaction at 0·58 MeV deuteron energy, and found a pronounced stripping contribution to the reaction cross section even at this low energy.

The purpose of the present investigation is to determine the relative importance of stripping and compound nucleus formation at 1·08 MeV bombarding energy and to determine the angular momentum of the protons captured by the stripping process.

§ 2. Experimental Conditions

The deuterons were accelerated in the Van de Graaff generator at the University of Bergen. A ^{10}B target supplied from the Atomic Energy Research Establishment, Harwell, England was used. The target thickness was approximately 80 kev for 1·08 MeV deuterons. Ilford C2 Nuclear Research Plates with emulsion thickness 200 μ were used. The neutrons entered the plates at grazing incidence. They were arranged in such a way that they covered the range of the angle $0° < \theta < 160°$, θ being

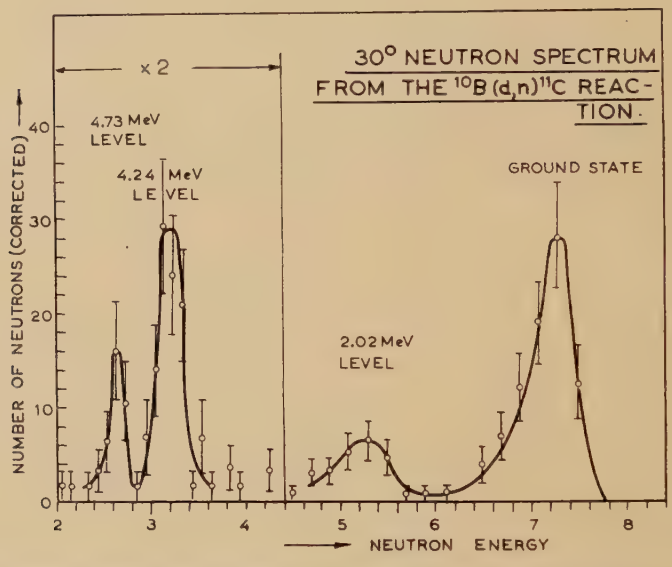
† Communicated by the Author.

the angle between deuteron and neutron directions. The distance between the target and the scanned areas was 120–130 mm, and independent of θ. The total deuteron charge on the target was 35μ A-hours, which gave a suitable density of recoil proton tracks in the emulsions. An improved photographic method described by Gibson (1953) was used, thus simplifying the measurements and calculations.

§ 3. Neutron Spectra

Neutron energy spectra have been measured at the following angles : 0°, 15°, 30°, 53°, 75°, 90°, 105° and 135°. Figure 1 shows the 30°-spectrum after correction for the variation with energy of the neutron-proton

Fig. 1



scattering cross section and for the escape of protons from the emulsion. The errors indicated are statistical errors only. The four neutron groups correspond to levels in ^{11}C at 0, $2{\cdot}02\pm0{\cdot}04$, $4{\cdot}24\pm0{\cdot}03$ and $4{\cdot}73\pm0{\cdot}03$ MeV. The first excited level seems to be a little higher than 1·90 MeV, as previously reported (Ajzenberg and Lauritsen 1955).

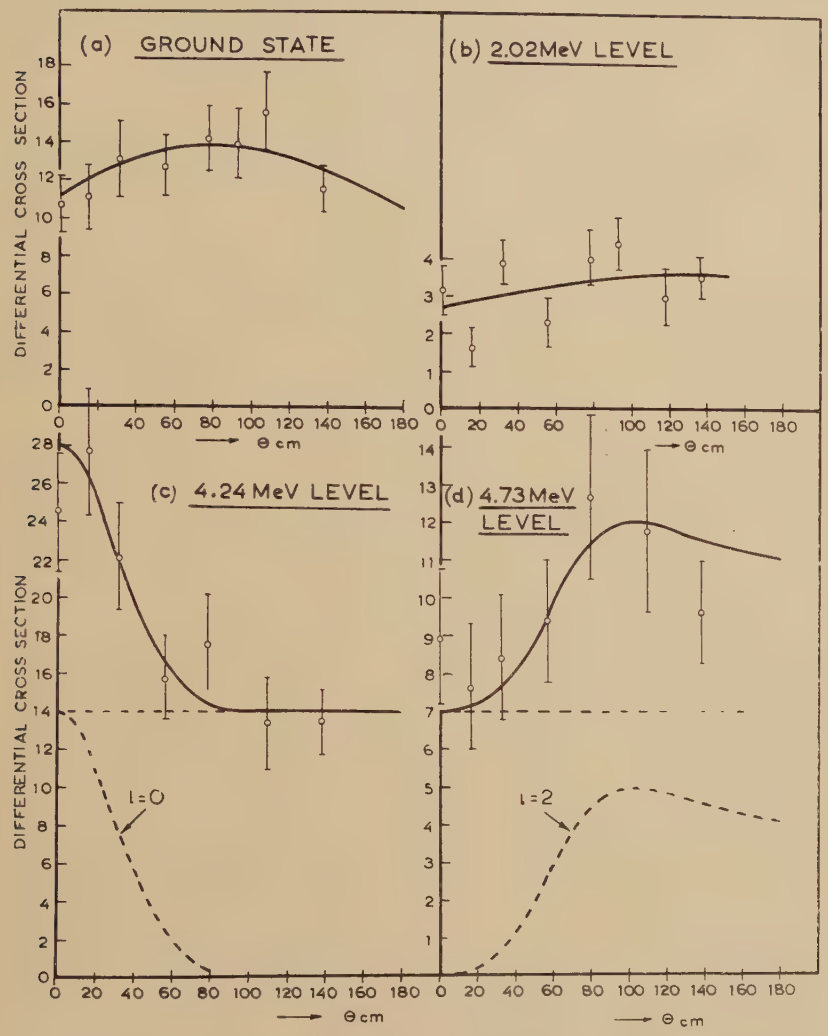
§ 4. Angular Distribution of the Neutrons

Burke *et al.* (1954) studied the angular distribution of neutrons from the ^{10}B (d, n) ^{11}C reaction at 1·06 MeV bombarding energy, but could not differentiate between the various neutron groups. It was, therefore, desirable to measure the angular distribution anew, using neutron detectors with energy discrimination.

The angular distributions obtained are shown in fig. 2. The distribution of the ground state neutrons in fig. 2 (*a*) is almost isotropic, thereby

indicating a large contribution from compound nucleus formation. This distribution is similar to that obtained at 0·85 MeV bombarding energy (Graue 1955). Figure 2 (*b*) shows the angular distribution of the neutrons corresponding to the 2·02 MeV level in ^{11}C. In spite of the poor statistics

Fig. 2



it can be said that the compound nucleus formation is responsible for a substantial part of the reaction yield. Figure 2 (*c*) and (*d*) shows the distributions of neutrons corresponding to levels in ^{11}C at 4·24 and 4·73 MeV, respectively. The experimental points can be adequately represented if a pure stripping curve is added to an isotropic distribution from compound nucleus formation. The theoretical stripping curves (stippled) were calculated by means of a formula from the non-coulomb stripping theory (Friedman and Tobocman 1953). The formula has been

rearranged in a form convenient for numerical calculations (Enge and Graue 1955). The nuclear radius of the target nucleus used in the calculation was $R=0{\cdot}43\times10^{-12}$ cm, which was computed from the expression $R=(0{\cdot}17+0{\cdot}122A^{1/3})+10^{-12}$ cm. The l-values found by comparison between the experimental angular distributions and the theoretical curves were 0 and 2 for the 4·24 MeV and 4·73 MeV level, respectively. These values are in agreement with previous measurements on the mirror-reaction ^{10}B (d, p) ^{11}B (Paris *et al.* 1954).

The large contribution from compound nucleus formation in all the angular distributions may be explained by the fact that our exposure is made at a deuteron energy where there is some indication of a very broad resonance (Ajzenberg and Lauritsen 1955).

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CVI. *Isotopic Spin Selection Rules VIII : Charge Independence and the Comparison of Isobaric Triplets*

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Abstract

The experimentally-determined positions of the first $T=1$ states of self-conjugate nuclei up to ^{40}Ca are compared with their positions calculated from the known ground state mass differences of the $T_z=+1$ and $T_z=0$ isobars. The Coulomb correction is made in a completely automatic manner that does not involve an assumed nuclear radius but uses the experimental energy difference in the corresponding mirror pair. Close correspondence between calculated and experimental positions is found, the former being on the average higher by 35 kev. If this difference is taken to reflect a departure from equality of the n–n and n–p singlet forces the latter is stronger by about 1% (square well). This is to be compared with the difference of 3% (square well) in the same sense found between p–p and n–p and n–p singlet forces from the zero energy scattering lengths.

§ 1. Introduction

In the series of which this paper is a part we have been chiefly concerned with testing the rule that if electric dipole radiation is to be freely emitted by a self-conjugate nucleus then the isotopic spin must change by one unit. The operating of the isotopic spin selection rules has in fact only been tested in such experiments on self-conjugate nuclei and in experiments in which $T=0$ particles provoke reactions and are produced as a result of reactions in self-conjugate nuclei. As is well known the isotopic spin selection rules which demand full charge independence (n–$n=n$–$p=p$–p) for their formulation, can for all practical purposes be replaced, when we deal with self-conjugate nuclei, by the charge parity selection rules (Kroll and Foldy 1952) which are based on the much less restrictive requirement of charge symmetry (n–$n=p$–p). So although we continue to talk of isotopic spin selection rules because we should like charge independence to be true we must not lose sight of the fact that all experiments so far performed, with one or two poorly-founded exceptions, are strictly relevant only for establishing charge parity selection rules and charge symmetry in which we are much more willing to believe *a priori* anyway. Our understanding of the situation in self-conjugate nuclei is now fairly good both experimentally and theoretically and, for example,

† Communicated by the Author.

paper VII of this series illustrates the general way in which the operation of the selection rules changes over a wide range of energy of excitation in a single nucleus (^{16}O). Charge symmetry, is then well established experimentally on the basis of the selection rules and on the basis of the well-known comparison of the level schemes of mirror nuclei. It is also very strongly to be expected on rather general theoretical grounds connected with the supposed origin of nuclear forces and so it is now appropriate to ask how we may extend our investigations to full charge independence which is comparatively poorly established experimentally and which cannot be expected theoretically with the confidence that we feel for charge symmetry.

There are three chief ways in which we may approach this problem of establishing charge independence from a study of light nuclei :

(i) We may study what we believe to be corresponding isobaric states in even nuclei of different T_z and see to what degree their dynamical properties are related in the way required by the isotopic spin being a good quantum number. Alternatively we may study the competing modes of decay of an excited system when the branching ratios for the emission of different particles to different members of an isobaric multiplet are related by, among other things, vector coupling coefficients in the isotopic spin. Both these methods suffer from the disadvantage that before comparison of the interesting quantities (reduced widths) may be made we must eliminate extraneous factors such as transmission coefficients. This is usually a very uncertain process, often depending rather sharply on what we take as the radius of the system. And often we do not know how to weight different orbital angular momenta of the product particles in the second type of investigation. Because of such considerations and because of the fact that the systems usually pass through rather highly excited states where the Coulomb perturbation may effectively relax the isotopic spin it seems unlikely that such measurements, although they are very interesting, will ever give quantitative information about the degree of charge independence. It is more probable that they will be useful as a means for studying the interplay of the Coulomb perturbation and the finite lifetime of the intermediate system in determing the ' effective ' purity of isotopic spin states.

(ii) We may study the properties of light nuclei of odd mass that we may make by bombarding a $T_z=+1$ target nucleus with protons, so forming both $T=\frac{1}{2}$ and $T=\frac{3}{2}$ states. Full isotopic spin rather than charge parity selection rules are then required to discuss the decay of these two classes of states. The two systems that suggest themselves are ^{14}C+p and ^{18}O+p ; already there is some suggestion that the first $T=\frac{3}{2}$ state of ^{15}N, characterized by a small reduced width for neutron emission to the ground state of ^{14}N, may have been found in the first of these systems (Bartholomew, Brown, Gove, Litherland and Paul 1955). This type of experiment must certainly be pressed hard for it offers rather good hope of seeing isotopic spin selection rules proper at work but, like the experiments in class (i), it suffers from passing through regions of rather high

excitation where, even if specifically nuclear charge independence were complete, the Coulomb forces might blur the definition of the isotopic spin beyond meaningfulness. The preliminary work on the $^{18}O+p$ system for example (unpublished work at the University of Minnesota and the Rice Institute) contains no sign of any isotopic spin selection rules at work ; here the Coulomb perturbation is expected to be more severe than in the former system.

The suggestion of Peaslee and Telegdi (1953) that the resonances of (γ,n) and (γ,t) processes be compared is a similar one but great technical difficulty is involved in the experiments.

(iii) The third possibility is to study rather more closely than has been done so far the evidence contained in the relative energies of the isobaric multiplets themselves. But for the Coulomb energy and the neutron–proton mass difference the members of, say, a $T=1$ triplet should have the same mass if charge independence holds and we neglect small effects. This would not be so if we had merely charge symmetry. It has been obvious for some time (e.g. Sherr, Muether and White 1949, Lauritsen 1952, Staehlin 1953, Inglis 1953, Ajzenberg and Lauritsen 1954, Burcham 1952, 1955) that there is at any rate approximate equivalence of mass in some cases but the very great uncertainties involved in extracting the Coulomb energy have hindered a serious examination of charge independence from this point of view. We shall now see how, if we are prepared to begin by accepting charge symmetry as complete, we may make quite a strong statement about charge independence based on the relative energies of the isobaric multiplets in nuclei of even mass.

§ 2. The Coulomb Energy Differences

There are several reasons why it is difficult actually to compute the Coulomb energy :

(i) The expression for the Coulomb energy contains the ' radius ' of the nucleus in the denominator. It is not very clear what ' radius ' means and how radii determined in different contexts are related ; they differ by 50% or more for the same nucleus.

(ii) The usual evaluation of the Coulomb energy neglects exchange effects which is certainly incorrect. It is not clear how they should most properly be allowed for (see e.g. Cooper and Henley 1953, Carlson and Talmi 1954, Peaslee 1954) but they can change the deduced radius by 20% or more.

(iii) The usual assumption that approximates the nucleus by a uniform sphere of charge is incorrect both from the point of view of the shell model and of direct experience with the scattering of fast electrons which shows that light nuclei are largely ' edge ' with only a small region of constant charge density if any at all (e.g. Fregeau and Hofstadter 1955). Owing to the mutual repulsion of the protons the form of the distribution will depend on Z for a given A.

(iv) Even if we knew the effective radial charge distribution we should still not know sufficient details of the wave functions to calculate the

Coulomb energy accurately. Nuclear forces favour the pairing of spins and so nuclei with even numbers of protons tend to have relatively greater Coulomb energies than those with odd Z. This was remarked on by Feenberg and Goerzel (1946) and shown up very clearly by Carlson and Talmi (1954). The effect is particularly striking in the p-shell.

(v) There is a complicated interaction between Coulomb and nuclear forces in which the former tend to increase the radius of the nucleus thereby changing the binding energy due to the latter. This effect will differ between members of an isobaric triplet because of the differing nuclear charges and so there will be a ' nuclear ' contribution to the ' Coulomb ' energy differences which is not taken into account in the conventional calculation which uses a constant radius parameter ' r_0 '.

Because of these various effects and uncertainties it is clear that we cannot accurately calculate the required differences between the Coulomb energies of the members of the isobaric triplets. We may find an empirical recipe in terms of parameters for the conventional ' uniform sphere ' calculation but even apart from the fundamentally unsatisfactory nature of this approach we could not look to it for better accuracy than several hundred kilovolts because of the finer effects we have listed.

If we are now prepared to assume the charge symmetry of nuclear forces we can do much better than this. Suppose we wish to discuss the lowest $T=1$ triplet $T_z=+1,\ 0,\ -1$ ($Z-1,\ Z,\ Z+1$) of mass A ($=2Z$). $Z-1$ and Z of mass $A-1$ are now mirror nuclei and their mass difference, corrected for the n–p mass difference, is due to the difference in ' Coulomb ' energies of the two nuclei where in ' Coulomb ' we include effects such as (v) above. We now multiply this energy by $(1-1/A)^{1/3}$ to allow for the change in radius on adding another particle and suppose that it represents the ' Coulomb ' energy difference between the $T_z=+1$ and $T_z=0$ members of the triplet. This procedure eliminates most of the difficulties mentioned above and has the advantage of being completely automatic, with no parameters to be manipulated to secure ' best fit '. The question of the radius and its definition are never raised. A particular advantage is that we may hope that we are allowing for the pairing effects in Z since the even–odd character of the charges is the same in the even A nuclei in which we are interested and in the mirror nuclei that we use for comparison. We are also automatically allowing for the various effects listed above which amount to the statement that ' r_0 ' depends on Z for a fixed A. The most serious defect of the method is probably that it makes no allowance for the difference in symmetry properties of the protons in the A and $A-1$ systems. We must hope that the influence of these differences is small.

Further lack of confidence may come from a comparison of the level structures of mirror nuclei where levels get out of step in the two members by as much as a few hundred kilovolts at excitations of a few MEV. To some degree this may be seen to be associated with the Thomas shift (see later) but it is also doubtless due in part to the Coulomb energy depending on the level we deal with. When the states concerned differ

considerably in structure from the ground states we must anticipate that they have different Coulomb energies from the ground states. In this connection our present procedure is not so bad as it looks since the lowest $T=1$ state of a $T_z=0$ nucleus is closely related to the ground state even though the $T=0$ levels at the same excitation as it may not be. Nor are the levels at the same exictation in the mirror $A-1$ nuclei necessarily closely related to their own ground states so the relative displacements in the $A-1$ nuclei that we have commented on as discouraging are not necessarily relevant.

A similar method to the present one has been used by Inglis (1953).

§ 3. Experimental Results

Our procedure is now to extract from experiment the energy difference between the mirror $A-1$ nuclei and use it, corrected by the small factor $(1-1/A)^{1/3}$, together with further experimental knowledge about the mass difference between the ground state of the $T_z=+1$ member of the A triplet and that of the $T_z=0$ member to predict the energy above the ground state of the $T_z=0$ nucleus at which we expect to find the first $T=1$ state if we have full charge independence. Call this energy $E_{T=1}$ (calc.). From experiment we now find the actual excitation energy $E_{T=1}$ (exp.) of this state and hence the difference $\Delta=E_{T=1}$ (calc.)$-E_{T=1}$ (exp.) between calculated and experimental positions. If no other factors entered into consideration Δ would now be a measure of the departure of nuclear forces from charge independence.

The two tables present the material available as far as ^{40}Ca.

Table 1 lists the energy differences between members of the mass $A-1$ mirror pair (allowing for the n–p mass difference). We also list the energy difference between the ground states of the $T_z=+1$ and $T_z=0$ members of the mass A triplet (allowing for the n–p mass difference); here a positive difference means that the $T_z=+1$ member is the heavier. For the cases of $A=26$ and 38 we do not need this latter information and although it may be known we do not list it; the measurement of energy difference in the mass 26 and 38 nuclei refers directly to the $T=1$ states of ^{26}Al and ^{38}K as is signified by the subscripts. For masses 10 and 14 it is also useful to know the $T_z=-1$ to $T_z=0$ mass difference so we list this as well.

Table 2 lists as $E_{T=1}$ (calc.) the calculated excitation energy of the first $T=1$ state in $T_z=0$ using the data of table 1 and the automatic method described above. If the energy of the $T_z=-1$ member of the triplet were accurately known we could calculate the expected position of the $T=1$ state in $T_z=0$ without recourse to the $A-1$ mirror pair, basing ourselves on the assumption of charge symmetry only. It is in fact known in several cases but usually only very inaccurately. In the two cases, namely $A=10$ and 14, the energy differences are well enough known from the $T_z=-1$ side (^{10}C and ^{14}O) to use this more satisfactory procedure. In just these two cases $E_{T=1}$ (calc.) refers to the energy computed

Table 1. Energy differences, allowing for n–p mass difference, between mirror nuclei of mass $A=1$ and between the ground states of the $T_z=+1$ (or -1) and $T_z=0$ nuclei of mass A (with the exception of ^{26}Al and ^{38}K—see text).

Mirror pair	Energy difference (MEV)	Ref.	Even mass nuclei	Energy difference (MEV)	Ref.
^{5}He—^{5}Li	0·89 ±0·20	(a)	^{6}He—^{6}Li	2·77 ±0·05	(j)
^{7}Li—^{7}Be	1·645±0·001	(b)	^{8}Li—^{8}Be	15·20 ±0·04	(b)
^{9}Be—^{9}B	1·852±0·002	(b)	^{10}Be—^{10}B	−0·226±0·003	(b)
			^{10}C—^{10}B	4·42 ±0·06	(b)
^{11}B—^{11}C	2·762±0·003	(b)	^{12}B—^{12}C	12·588±0·025	(b)
^{13}C—^{13}N	3·003±0·002	(b)	^{14}C—^{14}N	−0·627±0·001	(b)
			^{14}O—^{14}N	5·952±0·008	(j)
^{15}N—^{15}O	3·487±0·005	(c)	^{16}N—^{16}O	9·62 ±0·03	(j)
^{17}O—^{17}F	3·549±0·006	(d)	^{18}O—^{18}F	−2·453±0·009	(d)
^{19}F—^{19}Ne	4·027±0·008	(c)	^{20}F—^{20}Ne	6·261±0·012	(k)
^{21}Ne—^{21}Na	4·30 ±0·03	(d)	^{22}Ne—^{22}Na	−3·623±0·006	(d)
^{23}Na—^{23}Mg	4·841±0·010	(c)	^{24}Na—^{24}Mg	4·730±0·005	(l)
^{25}Mg—^{25}Al	5·084±0·025	(c)	^{26}Mg—^{26}Al$_{T=1}$	−5·006±0·010	(c)
^{27}Al—^{27}Si	5·584±0·010	(c)	^{28}Al—^{28}Si	3·868±0·011	(d)
^{29}Si—^{29}P	5·749±0·010	(e)	^{30}Si—^{30}P	−5·08 ±0·04	(m), (n)
^{31}P—^{31}S	6·00 ±0·10	(f)	^{32}P—^{32}S	0·925±0·004	(d)
^{33}S—^{33}Cl	6·34 ±0·12	(g)	^{34}S—^{34}Cl	−6·30 ±0·03	(n)
^{35}Cl—^{35}A	6·82 ±0·17	(h)	^{36}Cl—^{36}A	−0·068±0·005	(o)
^{37}A—^{37}K	6·78 ±0·12	(g)	^{38}A—^{38}K$_{T=1}$	−6·37 ±0·13	(p)
^{39}K—^{39}Ca	7·5 ±0·1	(i)	^{40}K—^{40}Ca	0·543±0·015	(o)

(a) This figure is the mean of two. One derives from the observed ^{5}He—^{5}Li mass difference—see Ajzenberg and Lauritsen (ref. (j) below). The other comes from assuming that the even–odd alternations in r_0 observed by Carlson and Talmi (1954) between $A=7$ and 9, and $A=11$ and 13 hold also between $A=3$ and 5 which enables us to compute the energy difference according to their recipe.

(b) C. W. Li, W. Whaling, W. A. Fowler, and C. C. Lauritsen, 1951, *Phys. Rev.*, **83**, 512.

(c) J. D. Kington, J. K. Bair, H. O. Cohn, and H. B. Willard, 1955, *Phys. Rev.*, **99**, 1393.

(d) C. W. Li, 1952, *Phys. Rev.*, **88**, 1038.

(e) H. Roderick, and C. Wong, 1953, *Phys. Rev.*, **92**, 204.

(f) F. J. Boley, and D. J. Zaffarano, 1951, *Phys. Rev.*, **84**, 1059 who measured ^{31}S(β^+)^{31}P, also R. N. H. Haslam, R. G. Summers-Gill, and E. H. Crosby, 1952, *Canad. J. Phys.*, **30**, 257 who measured the threshold for ^{32}S(γ, n)^{31}S from which, together with the ^{32}S and ^{31}P mass defects, the same datum may be deduced.

(g) No reliable data exist here. This energy difference is computed from the semi-empirical formula for Peaslee (1954) for mirror nuclei and the probable error taken from his estimate of its reliability.

(h) The old concordant but unreliable cloud chamber data which have been shown to be systematically too low have been discarded and this estimate follows (g).

(i) R. G. Summers-Gill, R. N. H. Haslam, and L. Katz, 1953, *Canad. J. Phys.*, **31**, 70 measured the threshold for ^{40}Ca(γ, n)^{39}Ca which combines with the mass defects of ^{40}Ca and ^{39}K to give the required datum.

(j) F. Ajzenberg, and T. Lauritsen, 1955, *Rev. Mod. Phys.*, **27**, 77.

(k) D. E. Alburger, 1952, *Phys. Rev.*, **88**, 1257.
C. Wong, 1954, *Phys. Rev.*, **95**, 765.

(l) K. Siegbahn, 1946, *Phys. Rev.*, **70**, 127.
A. Hedgran, and D. Lind., 1952, *Ark. Fys.*, **5**, 177.

(m) D. M. Van Patter, and W. W. Buechner, 1952, *Phys. Rev.*, **87**, 51.
C. E. Mandeville, C. P. Swan, S. D. Chatterjee, and D. M. Van Patter, 1952, *Phys. Rev.*, **85**, 193.

(n) D. Green, and J. R. Richardson, 1954, *Phys. Rev.*, **96**, 858.

(o) L. Feldman, and C. S. Wu, 1952, *Phys. Rev.*, **87**, 1091.

Table 2. Calculated and Experimental Excitations of the First $T=1$ state of Self-conjugate Nuclei and the Differences $\Delta = E_{T=1}$ (calc.) $- E_{T=1}$ (exp.).

Nucleus	$E_{T=1}$ (calc.) (MEV)	$E_{T=1}$ (exp.) (MEV)	Δ (MEV)	Ref.
^{6}Li	3·61 ±0·2	3·57 ±0·01	0·04 ±0·2	(q)
^{8}Be	16·77 ±0·04	16·72 ±0·01	0·05 ±0·04	(r)
^{10}B	1·775±0·027	1·739±0·003	0·036±0·027	(q)
^{12}C	15·27 ±0·03	15·10 ±0·02	0·17 ±0·04	(s)
^{14}N	2·336±0·004	2·313±0·004	0·023±0·006	(q)
^{16}O	13·08 ±0·03	12·95 ±0·01	0·13 ±0·03	(t)
^{18}F	1·029±0·009	1·075±0·010	−0·046±0·014	(u)
^{20}Ne	10·221±0·015	unknown	—	—
^{22}Na	0·61 ±0·03	near	near	
		0·592±0·003	0·02 ±0·03	(v)
^{24}Mg	9·504±0·012	9·47 ±0·10	0·03 ±0·10	(w)
^{26}Al	5·018±0·025−E	5·006±0·010−E	0·012±0·027	(x)
^{28}Si	9·384±0·017	9·27 ±0·11	0·11 ±0·11	(w), (y)
^{30}P	0·61 ±0·04	0·688±0·007	−0·08 ±0·04	(z)
^{32}S	6·86 ±0·10	7·02 ±0·05	−0·16 ±0·12	(w)
^{34}Cl	−0·02 ±0·12	0	−0·02 ±0·12	(a′)
^{36}A	6·06 ±0·06	unknown	—	—
^{38}K	6·72 ±0·12−E	6·37 ±0·13−E	0·35 ±0·18	(b′)
^{40}Ca	7·9 ±0·1	unknown	—	—

(p) We use the measurement of Boley and Zaffarano (f) above which, in view of the results of P. STAEHLIN, 1953, *Helv. Phys. Acta*, **26**, 691 we re-interpret as referring to the $T=1$ state of ^{38}K. This entry is therefore subject to some doubt, particularly since ^{37}K to which it was originally supposed the measurements referred may well have a lifetime and positron end point close to those of ^{38}K$_{T=1}$.

(q) First $T=1$ state in $T_z=0$ well established—see reference (j).

(r) The evidence for the first $T=1$ state being around 17 MEV was given by M. GELL-MANN and V. L. TELEGDI, 1953, *Phys. Rev.*, **91**, 169, on the basis of its formation by alpha-particle emission from ^{12}C following E1 absorption. F. K. GOWARD, and J. J. WILKINS, 1955, *Proc. Roy. Soc.* A, **228**, 376, propose the same assignment and find more than one $T=1$ level in this region, the lowest well-established one being at 16·8±0·2 MEV, our quoted figure comes from T. W. BONNER, and C. F. COOK, 1954, *Phys. Rev.*, **96**, 122. The assignment 2+ for this level from the ^{12}C(γ, α)^{8}Be work is consistent with the known properties of ^{8}Li (j).

(s) That this first $T=1$ state is around 16 MEV is suggested by C. A. HSIAO, and V. L. TELEGDI, 1953, *Phys. Rev.*, **90**, 494, on the basis of its frequent production following E1 absorption in ^{16}O(γ, α)^{12}C. Many workers (see (j)) have observed that a level at 15·10 MEV (this energy comes from C. F. Cook, J. Marion and T. W. Bonner (j)) emits γ-rays to the ground state in successful competition with alpha-particles. This strongly suggests $T=1$ and the emission to the ground state is consistent with the 1+ assignment usually given to ^{12}B.

(t) This assignment is not sure. ^{16}N is 2− so we seek a $T=1$ 2− state in ^{16}O around 13 MEV. Two 2− states are known, at 12·51 and 12·95 MEV. Both have very large reduced proton widths for d-wave emission to the ground state of ^{15}N as we should expect if they were simple states such as the ground state of ^{16}N is presumed to be. This suggests that they may be the $T=0$ and $T=1$ linkages of a $d_{5/2}$ proton to the ground state of ^{15}N just as we suppose the ground state of ^{16}N to be the linkage of a $d_{5/2}$ neutron (neglecting configuration mixing in both cases) to the ground state of ^{15}N. The lower ^{16}O state has a reduced alpha-particle width for emission to the 4·43 MEV state of ^{12}C of 80% of the single-particle value, the upper state has a corresponding width of 4%. So if we are to choose our $T=1$ state from these two we choose that at 12·95 MEV. This choice is supported by the associated Δ value of 130 KEV which would become much bigger than any other example if we chose the lower state as $T=1$. We may note as a comment on the large alpha-particle width of our supposed first $T=1$ 2− state that this $T=1$ 1− state also shows an

alpha-particle width of 5% of the single particle value. It seems that the high density of $T=0$ states in this region is relaxing the definition of isotopic spin rather considerably; note in particular that if we are correct in supposing that the 12·51 and 12·95 MEV 2— levels are closely-related $T=0$ and $T=1$ states we might expect them to contaminate each other rather effectively—see D. H. WILKINSON, and A. B. CLEGG, 1956, *Phil. Mag.*, **1**, 91, part VI of this series. We must admit that this identification is as yet unsure; it is clear from the density of levels near the ground state of ^{16}N that all $T=1$ states in this neighbourhood in ^{16}O have not yet turned up.

(u) Only this state is known within 800 kev of the expected energy. Its properties are consistent with the 0+ required for its being $T=1$ (P. C. PRICE, 1955, *Proc. Phys. Soc.* A, **68,** 553). It is formed in ^{20}Ne(d, α)^{18}F and this has always stood against its being recognized as of $T=1$. However the total excitation in the intermediate ^{22}Na system in these latter experiments is only 18·5 MEV at which energy we may still expect some resonance structure to persist. Under these conditions we must be prepared to find severe breakdown of the isotopic spin selection rules (A. M. LANE, and R. G. THOMAS, 1956, *Rev. Mod. Phys.* (in press); D. H. WILKINSON, *Phil. Mag.* (paper VII of this series); C. P. BROWNE (private communication)) and until another and more acceptable state is found in this immediate energy region it seems that we may tentatively look on the present one as of $T=1$.

(v) N. P. HEYDENBURG, and G. M. TEMMER, 1954, *Phys. Rev.*, **94,** 1252, have observed a gamma-ray of 592±3 kev which they supposed to come from the $0+T=1$ state. However they also observed that its half-life against presumed M3 radiation to the 3+ ground state was less than 10^{-2} sec. about ten times the single-particle speed which makes the identification doubtful. S. VEGORS, and P. AXEL (private communication) have sought an isomeric state in ^{22}Na following ^{23}Na(γ, n)^{22}Na and are able with good probability to place an upper limit of 10 μsec on the lifetime. It therefore appears probable that the 592 kev state is not 0+ but perhaps 1+ with the true $0+T=1$ state lying closely above it, being fed but with its M1 radiation being of low enough energy to have passed unnoticed in Heydenburg and Temmer's work. Again C. P. BROWNE, and W. C. COBB, 1955, *Bull. Amer. Phys. Soc.*, No. 3, p. 44, find the 592 kev state strongly excited in ^{24}Mg(d, α)^{22}Na which may tell against its being of $T=1$. Here the arguments of (u) about the production of $T=1$ states in such reactions again apply although we should only use them to excuse a $T=1$ state's appearance rather than to lead ourselves to expect it (c.f. the similar reaction ^{28}Si(d, α)^{26}Al where the $T=1$ state is not produced—C. P. BROWNE, 1954, *Phys. Rev.*, **95,** 860, see also R. W. KAVANAGH, W. R. MILLS, and R. SHERR, 1955, *Phys. Rev.*, **97,** 248). We therefore consider it likely that the $T=1$ state is at a slightly higher excitation than the 592 kev state but close enough for the cascade radiation to have been missed.

(w) N. W. GLASS, and J. R. RICHARDSON, 1955, *Phys. Rev.*, **98,** 1251, have studied the positron emitters ^{24}Al, ^{28}P and ^{32}Cl and find in each of them evidence for allowed favoured transitions such as are expected from them to the analogue $T=1$ level in the $T_z=0$ nucleus. The quoted level positions then come from the appropriate compounding of measured gamma-ray energies (averaged with another datum for ^{28}Si—see (y)).

(x) The situation in ^{26}Al is now quite well understood (see M. J. LAUBITZ, 1955, *Proc. Phys. Soc.* A, **68,** 1033 for a summary). The experimental datum here of greatest accuracy is a measurement of the Q-value for ^{26}Mg(p, n)^{26}Al$_{T=1}$ direct to the first $T=1$ state of ^{26}Al (ref. (c) above) so we do not need to bother with determining the position of this level relative to the ground state of ^{26}Al. In the entry of table 2 E is the energy difference, corrected for the n–p mass difference, between the ground state of ^{26}Mg and that of ^{26}Al.

(y) J. M. CALVERT, A. A. JAFFE, A. E. LITHERLAND, and E. E. MASLIN, 1955, *Proc. Phys. Soc.* A, **68,** 1008, have measured the stripping properties of a level in ^{28}Si at 9·16±0·17 MEV seen in the reaction ^{27}Al(d, n)^{28}Si and find a close correspondence with those of the ground state of ^{28}Al seen in ^{27}Al(d, p)^{28}Al. This leads them to suggest this level as the first $T=1$ state of ^{28}Si. The energy is in good agreement with the 9·37±0·15 MEV deriving from the decay of ^{28}P—see (w)— and we have combined these values.

(z) Again the only justification for regarding this level as of $T=1$ is that it is the only known level within 600 kev of the expected energy. P. M. ENDT, J. C. KLUYER, and C. VAN DER LEUN, 1954, *Phys. Rev.*, **95,** 580—see also reference (w).

(a′) D. C. PEASLEE, 1953, *Nuovo Cimento*, **10,** 1349.

(b′) STAEHLIN (1953)—main references. The calculated entry, hence Δ, is subject to some doubt—see reference (p). In this entry of table 2 E is the energy difference, corrected for the n–p mass difference, between the ground state of ^{38}A and that of ^{38}K.

using the data from $T_z=\pm 1$ and without reference to the data from the mirror pair of table 1. In making this computation we have followed the method of Carlson and Talmi (1954) and estimated the (relative) Coulomb corrections between $T_z=+1$ and $T_z=0$ and between $T_z=-1$ and $T_z=0$ using LS wave-functions; these are the more valid approximation in light nuclei. It is now interesting to check these results for ^{10}B and ^{14}N against what we should have obtained using the method based on the mirror transition which we have used in all other cases. In the case of ^{14}N the agreement is good and we get $2{\cdot}302\pm 0{\cdot}003$ MEV for $E_{T=1}$ (calc.) from the mirror pair as against $2{\cdot}336\pm 0{\cdot}004$ MEV from the better $T_z=\pm 1$ method. This is good confirmation of the validity of our general method of using the mirror pair. In the case of ^{10}B we must expect a bad estimate from using the mirror pair because the proton-rich member, ^{9}B, is actually unstable against proton emission. This will cause its mass to be depressed by the Thomas (1952) shift representing the fact that the loose proton can spend much of its time at the edge of the nucleus where its Coulomb potential energy is low. So we expect the ^{9}B–^{9}Be entry of table 1 to be too low if we use the table 1 data. In fact this is so and $E_{T=1}$ (calc.) by the mirror pair method would have been 1·56 MEV as against 1·78 MEV of the better method and the 1·74 MEV of experiment. It is fortunate that in this, the case where we would have least confidence in the mirror pair method (with the exception of $A-1=5$) we have the better $T_z=\pm 1$ method to rely on.

The experimental $T=1$ assignments are discussed in the table 2 references.

§ 4. Discussion

The entry Δ of table 2 shows that when we make the Coulomb correction in this automatic fashion we do indeed find extremely close agreement between the calculated and experimental positions of the first $T=1$ states of $T_z=0$. In most cases the value and probable error in Δ fit together so well that it is difficult to see that there is any significant discrepancy at all. Only in ^{12}C and ^{16}O do we seem to find substantial disagreement. These cases may be somewhat special in that the excitation energy involved in $T_z=0$ is considerable and we are in a region of quite closely spaced $T=0$ states (not so for the otherwise-comparable ^{8}Be) whose density is increasing with increasing excitation. Under these circumstances we may expect considerable isotopic spin mixing with these $T=0$ states, and so displacement of the $T=1$ level by the usual mechanism of configuration interaction. This will tend to be towards lower energies because of the greater abundance of mixing (and repelling) states above than below. This is to some degree confirmed by the observed high impurity of the $T=1$ states in this region of ^{16}O—see reference (t) of the table 2 References—although we should comment that in contrast the purity in the corresponding case of ^{12}C appears to be high enough to allow gamma-ray emission to compete with alpha-particle decay—reference (s) of the table 2 References. In ^{16}O a special factor may be

seen at work to produce a displacement in the observed direction. We take the Coulomb shift from $^{15}N-^{15}O$ where in the latter body we have 6 1p protons. However in the $T=1$ state of ^{16}O we have one proton removed from the 1p to the 1d shell. This means that it sits somewhat nearer the edge of the nucleus and so we have a smaller Coulomb shift between the $T=1$ state of ^{16}O and the ground state of ^{16}N than we calculate from $A=15$. A rough estimate of the quantitative importance of this effect, using shell-model wave functions for the last proton and a uniform sphere for the rest of the nucleus, gives 40 kev. However no such argument can be made for ^{12}C. Another factor tending to produce a positive Δ for these highly-excited $T=1$ states is the Thomas shift. These states are either unstable against proton emission (^{16}O) or only just stable (^{8}Be, ^{12}C) and so we must anticipate a downward shift as we noticed in our discussion of ^{9}B in § 2. This is in the right sense but it appears to be substantially too small to account for the observed Δ in these cases.

A significant tendency is for positive Δ values at any rate in the p-shell. There seems no general reason for this unless it be that the proton binding energy tends to be less for the $T=1$ state of $T_z=0$ than in the mirror nucleus with an attendant relative Thomas shift. This would have to be surprisingly large to account for the observed Δ values.

There are two small effects which go in the wrong sense—tend to raise the $T=1$ state of $T_z=0$. As we noted in § 1 part of the 'Coulomb' energy expresses the distortion of the nucleus by the Coulomb forces with attendant change in the binding by the specifically nuclear forces. This is allowed for by our use of data from the mirror pair. However in the even A nuclei the specifically-nuclear forces tend to be relatively stronger because of the extra neutron so we should expect the even A nuclei to be relatively smaller (smaller effective 'r_0') and so have a bigger Coulomb shift. The second effect is closely related to this and is again that the even A nuclei will tend to be relatively smaller—because of the pairing effect of an even number of particles. An attendant effect to this last one goes sometimes one way sometimes the other and its failure to show up as an alternation in Δ may indicate that these fine effects are indeed as small as we should expect them to be. It is that when $T_z=0$ is even-even, $T_z=+1$ is odd–odd and vice versa. In the first case we should expect $T_z=0$ to be relatively smaller and so Δ to be decreased; in the second case Δ should be increased by this effect. As we see from table 2, if anything even–even $T_z=0$ have the larger Δ values.

Beyond the p-shell the probable errors are larger and no significant trend of Δ can be discerned.

We may sum up this discussion by saying that, although there seems to be an unexplained tendency towards positive values, Δ is on the whole remarkably small. If we award a weight of 1 to Δ values with probable errors of 50 kev or less and divide the weight by 4 for each range of doubling of the probable error we find a mean Δ value for all nuclei listed in table 2 of +35 kev. It seems then that we can state that the

average discrepancy between calculated and expected positions of the $T=1$ states in less than 50 kev. It is significantly positive.

§ 5. Charge Independence

Our Δ values are very small and so imply that the n–p bond of $T_z=0$ is closely equivalent to the n–n bond of $T_z=+1$. These nuclei differ only in the single bond so that if the n–p force is not the same as the n–n there will be a gross shift in both nuclear masses due to all the n–n (or p–p) and n–p bonds differing but the *relative* shift with which we are concerned here is due to the single bond alone. If we regard the departure from complete charge independence as a perturbation, presuming charge symmetry, we expect a relative shift of the order

$$\Delta \sim \int \psi^* \Delta S \psi d\tau$$

where ΔS is the difference in strength between the n–n and n–p (chiefly ^{31}S) interactions or

$$\Delta \sim \frac{\Delta S}{S} VP$$

where $\Delta S/S$ is the fractional departure from charge independence, V is the depth of the ^{31}S interactions (say 10 Mev for a square-well range of $a = 2{\cdot}8 \times 10^{-13}$cm) while P is the fraction of the time for which the two nucleons are within range of each other's attraction ($\sim(a/R)^3$ for a big nucleus). If we insert our value of 35 kev for Δ for light nuclei with say $P \sim \frac{1}{2}-\frac{1}{3}$ we find that in the square-well approximation the n–p interaction is stronger than the n–n by $0{\cdot}7-1\%$. Allowing for the very rough nature of this estimate it still seems that these data on the positions of the first $T=1$ states cannot admit of more than a very few per cent departure from charge independence.

It is interesting to compare this tentative conclusion that n–p forces may be stronger than n–n by about 1% with the results deriving from n–p and p–p singlet scattering. The two singlet scattering lengths are very significantly different but, since both are very large, reflecting the very small 'energy of the singlet deuteron', the difference in n–p and p–p well strengths is small. Again in the square well approximation the n–p strength is greater than the p–p by 3·3% after allowance for the Coulomb effect (Jackson and Blatt 1950) to be compared with our 1% for the excess of n–p force over n–n. The effects are seen to have the same sign and to be of the same order. Schwinger (1950) has pointed out that the difference between the n–p and p–p forces may be semi-quantitatively accounted for as due to the different magnetic interactions of the two systems, an explanation that would leave the specifically-nuclear charge independence complete. We should then expect just about the same difference between the n–p and n–n potentials as between the n–p and p–p. If we omit the relatively small orbit–orbit contribution to the

p–p magnetic potential the three magnetic contributions stand in the ratio :

$$\begin{array}{cccccc} & V_{\text{n-n}}^{\text{mag}} & & V_{\text{p-p}}^{\text{mag}} & & V_{\text{n-p}}^{\text{mag}} \\ = & \mu_{\text{n}}{}^2 & : & \mu_{\text{p}}{}^2-\mu_{\text{p}}+\tfrac{1}{2} & : & \mu_{\text{n}}(\mu_{\text{p}}-\tfrac{1}{2}) \\ & 3{\cdot}66 & : & 5{\cdot}51 & : & -4{\cdot}39 \end{array}$$

The μ's are the magnetic moments in nuclear magnetons. It is not at all clear that this magnetic explanation for the difference between n–p and p–p scattering lengths is adequate since Schwinger found that a square well gave unsatisfactory results and a Yukawa-type potential was needed. In view of the current favouring of a repulsive core to the nucleon–nucleon potential it is unlikely that the satisfactory agreement of the Yukawa potential can be maintained (Salpeter 1953). But in any case our present result on the n–n versus n–p forces is consistent with the information from scattering lengths and the basic assumption of charge symmetry. It is also consistent with the much less accurate information relating to comparison of the n–p and p–p effective ranges in the singlet state (see e.g. Storrs and Frisch 1954).

If we are to invoke magnetic forces in our comparison of the n–n and n–p interactions from the isobar structure we must also consider the magnetic contribution to the mass of the isobars due for example to the coupling between the magnetic moments of the nucleons and those of the mirror nuclei to which they are added or, more properly, to the moments of the various parents in $A-1$. These effects are of comparable importance with the nucleon–nucleon magnetic energies but vary in sign from case to case and so tend to cancel out in our evaluation of the mean Δ.

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CVII. *The Nuclear Surface*

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Abstract

The basic ideas of Brueckner's self-consistent nuclear model are applied in a simplified, approximate form to the case of a finite nucleus. It is shown that it is possible to reconcile the observed values of surface energy and surface thickness, to explain the greater extension of the nuclear potential compared with the charge distribution, and also to fit the well-depth of the optical model for nucleon scattering.

§ 1. Introduction

A difficulty has recently been pointed out by Swiatecki (1955 b) in reconciling the value of the nuclear surface energy with the measurements of surface thickness obtained from electron scattering experiments. This problem is reinvestigated here in the light of the probable existence of a velocity-dependent mean potential, as suggseted by Brueckner *et al.* (1954)‡ and by Johnson and Teller (1955).

In the self-consistent model of Brueckner, whose ideas are followed here, nuclear saturation is achieved by the reduction in the attractive force between pairs of nucleons as their relative velocity increases. In the idealized case of indefinitely extended nuclear matter, the effective potential between pairs of nucleons is completely described in terms of their relative momenta ; in the case of a finite system the effective potential must depend upon both momenta and coordinates, and as has been pointed out by Bethe (1956), the mean central potential should then properly be taken as a potential matrix, diagonal neither in co-ordinate nor momentum space.

To provide a simple model for the present discussion it has here been assumed that the effective interaction between pairs of nucleons may be described by delta-functions of position and differential operators (to express momentum dependence) ; the resultant mean potential is then a sum of products of functions of position with momentum operators. With these assumptions a relatively simple set of self-consistent equations may be written down for the individual particle states of the assumed model wave-function.

† Communicated by Dr. B. H. Flowers.
‡ The principal reference for this paper is Brueckner (1954 b).

The problem of determining self-consistent scattering parameters from an assumed inter-nucleon potential is not discussed here ; a simple form for the scattering amplitude is taken with parameters adjusted to fit some of the experimental data.

An estimate of the nature of the solution of these self-consistent equations is made by using the Thomas–Fermi approximation for the momentum distribution. This leads to certain definite results for the surface energy and for the shapes of the density distribution and of the mean potential well.

§ 2. The Model Equations (Infinite System)

In the case of infinitely extended nuclear matter the determinantal, model, wave function introduced by Brueckner is built out of individual plane-wave states, and Brueckner has shown how in good approximation the energy of the system may then be calculated as the expectation value of kinetic energy plus an effective potential proportional to the forward scattering amplitude between the plane wave states. In a complete calculation, self-consistent *ab initio*, this scattering has to be calculated from the assumed two-body potential, and evaluated for particles inside nuclear matter, subject therefore to the mean potential field and to the exclusion principle. This full problem is not considered in the present paper ; only the form of the theory is used, slightly generalized to apply to a finite system, and with the constants in the scattering amplitude adjusted to give the correct volume saturation effects.

In this model saturation is achieved through the velocity dependence of the scattering amplitude. For a determinantal wave function built out of plane-wave states, with momenta $\mathbf{k}_i$ and normalized in a volume V, the effective interaction energy between a pair of nucleons with momenta $\mathbf{k}_i$ and $\mathbf{k}_j$ is a function only of the relative momenta,

$$(1/V)t(\mathbf{k}_i-\mathbf{k}_j). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In Born approximation, and neglecting the effects of exchange, t is equal to the volume integral of the assumed two-body potential. In general the velocity-dependence of t arises in part from the contributions of the exchange integrals, and in part from the energy dependence of the scattering amplitudes, when calculated exactly.

On the average, for the momenta occurring within the nucleus, t is negative, expressing the binding of the nucleons ; saturation will obtain if t becomes more positive, sufficiently rapidly, with increasing momentum. A simple form that shows this behaviour is

$$t(k)=t_0+t_1k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with negative t_0 and positive t_1. This might be regarded either as the first two terms in the complete expansion of t in powers of k, or perhaps more realistically as an approximation describing the behaviour of t over the important momentum range ; in either case the parameters t_0, t_1 will be functions of the density in the self consistent calculation.

It will be convenient, for the subsequent consideration of the finite system, to introduce symbols ρ and τ to denote the particle and square of momentum densities; for plane waves in a volume V these are

$$\rho=(1/V)\,\Sigma_i 1\,; \qquad \tau=(1/V)\,\Sigma_i k_i^2. \quad . \quad . \quad . \quad (3)$$

Then the interaction energy density (of the infinite system) is

$$(1/V)^2{}_i\Sigma_{<j}\, t(\mathbf{k}_i-\mathbf{k}_j)=\tfrac{1}{2}t_0\rho^2+t_1\rho\tau. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Here the cross terms in $\mathbf{k}_i\cdot\mathbf{k}_j$ are assumed to vanish on summation.† Addition of the kinetic energy to this gives for the total energy density

$$H_\infty=(1/(2M)+t_1\rho)\tau+\tfrac{1}{2}t_o\rho^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the coefficient of τ may be interpreted in terms of an effective mass M^* defined by

$$1/(2M^*)=1/(2M)+t_1\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In this infinite system where the states are plane waves with momenta filling up a sphere (in momentum space), the kinetic energy is a known function of the density and with four nucleons to each state.

$$\tau=(3{\cdot}6)\rho^{5/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then the energy per particle $E=H/\rho$ is a known function of the density, and the equilibrium density and binding energy are obtained by minimizing $E(\rho)$.

§ 3. The Model Equations (Finite System)

The appropriate generalizations of these ideas to the case of finite system have been considered by Bethe (1956) and by Eden (1956). The suitability of any particular prescription for describing the model wave function depends ultimately upon an examination and evaluation of the correction terms that subsequently arise. As this analysis involves the relation of the scattering matrix to the potential it will not be discussed further here ; the form of the equation to be used will be the simplest possible generalization of those of the infinite system.

The scattering described by t is a local phenomenon ; roughly speaking $t(\mathbf{k})$ may be taken as the matrix element of a ' pseudopotential ' which is its Fourier transform, and if the approximation (2) is used this potential will be described by delta-functions and differential operators; the coefficient of $\mathbf{k}^2$ is a measure of the range of the potential. To analyes this more clearly it is desirable to consider the complete scattering matrix for the two nucleon system, regarding the nucleons as distinguishable ; suppose then that the scattering from relative momentum $\mathbf{k}$ to $\mathbf{k}'$ has the amplitude

$$(\mathbf{k}' \mid T \mid \mathbf{k}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where T is also an operator in spin and isotopic spin. Then the amplitude used in the description of the infinite system is

$$t(\mathbf{k})=(\mathbf{k}|\overline{T}|\mathbf{k})-(-\mathbf{k}|\overline{TP_\sigma P_\tau}|\mathbf{k}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

† It is not necessary to exclude the terms with $i-j$, because these will give no contribution when t properly includes the effects of exchange.

where P_σ, P_τ are the spin and isotopic spin exchange operators, and $\overline{T}$ denotes the average of T over spins and isotopic spins. $t(k)$ may be regarded as the diagonal element of

$$(\mathbf{k}'|t|\mathbf{k})=(\mathbf{k}'|\overline{T}|\mathbf{k})-(-\mathbf{k}'|\overline{TP_\sigma P_\tau}|\mathbf{k}) \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and it will be supposed that this can be represented approximately by

$$(\mathbf{k}'|t|\mathbf{k})=t_0+t_{1e}k^2+t_{10}\mathbf{k}'.\mathbf{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where t_{1e} and t_{10} are then the parts of t_1 associated with scattering in even and odd states respectively. This amplitude (11) is equal to the direct matrix element of the ' pseudopotential '

$$t_0\,(\mathbf{x}-\mathbf{x}')-\tfrac{1}{2}t_{1e}[\delta(\mathbf{x}-\mathbf{x}')\,(\nabla-\nabla')^2+(\nabla-\nabla')^2\,\delta(x-x')]$$
$$-t_{10}(\nabla-\nabla')\,.\,\delta(\mathbf{x}-\mathbf{x}')\,(\nabla-\nabla'). \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The finite system will be discussed by using (12) to evaluate the effective potential energy of a determinantal model wave function. The coefficients t_0, etc. may be functions of position; it will here be assumed that they are the same functions of the local density as for an infinite system. It will be shown below that it is not necessary for consistency to suppose that these coefficients vary greatly with density, so that the assumption is not obviously unreasonable ; further discussion would involve the (real) potential and the correction terms of the theory.

The model wave function is assumed to be a determinant of orthogonal (real) single-particle states $\phi_i(x)$. The local densities ρ and τ are defined by the natural generalizations of the definitions (3),

$$\rho=\Sigma_i[\phi_i(x)]^2\,; \qquad \tau=\Sigma_i[\nabla_i(x)]^2. \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The interaction energy of the system is the expectation value of T, eqn. (8), for the determinantal wave function, and this is equal to the expectation value of (12) for the simple product wave function of ϕ_i. The interaction energy density is easily evaluated in terms of ρ and τ, and is

$$\tfrac{1}{2}t_0\rho^2+t_1\rho\tau+t_{1e}[\tfrac{1}{2}\nabla(\rho)^2-\tfrac{1}{2}\rho^2\nabla\rho]-\tfrac{1}{4}t_{10}(\nabla\rho)^2. \quad . \quad . \quad . \quad (14)$$

If it is supposed that t_{1e} is a function only of ρ the term in $\nabla^2\rho$ can be transformed into a multiple of $(\nabla\rho)^2$, and the total energy density written as

$$H=[1/(2M^*)]\tau+\tfrac{1}{2}t_0\rho^2+\tfrac{1}{2}B(\nabla\rho)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

of which the first two terms are identical with the energy density of the infinite system. In particular if it is supposed that t_1 is constant, and the scattering only in even states, then

$$B=(3/2)t_1 \quad \text{and} \quad 1/(2M^*)=1/(2M)+\tfrac{2}{3}\rho B. \quad . \quad . \quad . \quad (16)$$

It is with the effects of the last term in the energy that this paper is chiefly concerned ; it has manifestly the form of a surface energy, and appears in the present treatment as a consequence of the higher momentum components needed in the wave function to describe the surface.

The statement of the problem in the form (15), leads immediately, by variation of the total energy, to a set of self consistent equations, of the Hartree type, for the best set of single particle wave functions $\phi_i(x)$.

The single particle states are the eigenstates of the single particle Hamiltonian

$$h=-\nabla\,[1/(2M^*)]\,.\,\nabla+U(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

with the mean central potential defined self consistently by

$$U(x)=\frac{d}{d\rho}\,(\tfrac{1}{2}t_0\rho^2)+\tau\,\frac{d}{d\rho}\left(\frac{1}{2M^*}\right)-B\nabla^2\rho-\tfrac{1}{2}\nabla B\,.\,\nabla\rho \quad . \quad . \quad (18)$$

M^* is defined by eqn. (6) and ρ and τ are functions of x given by (13).

§ 4. Further Approximations

The solution of the self consistent field eqns. (17) and (18) would require much computation ; to get a simple qualitative picture of their implications it is necessary to make further approximations. The obvious approximation in the present context is that of Thomas and Fermi, relating the local kinetic energy to the density ; this will be used here in the form of eqn. (7)

$$\tau=(3{\cdot}6)\rho^{5/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and with this substitution eqn. (15) leads to an ordinary differential equation for the density distribution.

To keep the algebra as simple as possible, a further approximation has been introduced to represent the volume energy terms. These are

$$H_\infty=\rho E(\rho)=\{1/(2M)+\rho t_1\}(3{\cdot}6)\rho^{5/3}+\tfrac{1}{2}t_0\rho^2 \quad . \quad . \quad . \quad . \quad (19)$$

and the minimum of $E(\rho)$ should give the density and volume binding energy per particle ; the experimental values of these quantities are taken as

$$\rho_0=(4\pi r_0^3/3)^{-1}, \quad \text{with} \quad r_0=1{\cdot}2\times10^{-13}\ \text{cm} \ ; \quad . \quad . \quad . \quad (20)$$
$$\epsilon_0=15\ \text{MeV}.$$

As mentioned in § 2 the coefficients must in general be functions of the density, but it is not in fact necessary to assume a strong dependence to achieve the saturation values given by (20). Again for simplicity, it will be assumed that t_1 is a constant, and the value that will be chosen later (from consideration of the surface effects) is such that

$$\rho_0 t_1=(0{\cdot}5)/(2M). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

Then the values (20) will be obtained at the minimum of $E(\rho)$ if

$$\begin{aligned} -E_0=E(\rho_0)&=\{1{\cdot}5/(2M)\}(3{\cdot}6\ \rho_0^{2/3})+\tfrac{1}{2}t_0\rho_0 \\ 0=\rho_0 E'(\rho_0)&=\{1{\cdot}5/(2M)\}(3{\cdot}6\rho_0^{2/3})+\tfrac{1}{2}t_0\rho_0+\tfrac{1}{2}t_0'\rho_0^2 \quad . \quad (22) \end{aligned}$$

where t_0' is the value of $dt_0/d\rho$ at $\rho=\rho_0$; and these equations then determine

$$\tfrac{1}{2}t_0\rho_0=-3{\cdot}1E_0 \ ; \qquad \tfrac{1}{2}t_0'=1{\cdot}0\,E_0. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

These values suggest that the variation of t_0 with density is not excessive.

As it is not the purpose of this paper to discuss the relation of t_0 to the internucleon potentials, a simple form has been chosen for $E(\rho)$ that satisfies the conditions (20) and also that $E(0)=0$; such is

$$E(\rho)=E_0\{(1-\rho/\rho)_0^2-1\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The form of $E(\rho)$ for low densities is not important in the sequel, and the most critical assumption hidden in the form (24) is the value of the compressibility coefficient; if this is defined, as by Feenberg (1947), as $9\rho^2 d^2E/d\rho^2$, its value is 270 MeV; in terms of t_0, as in eqn. (22), this value would imply a second derivative given by

$$\tfrac{1}{2}\rho_0^3 t_0'' = -0{\cdot}47E_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

The Hamiltonian density of the system is therefore taken as

$$H = \{1/(2M^*)\}\,(\tau - 3{\cdot}6\rho^{5/3}) + \rho E(\rho) + \tfrac{1}{2}B(\nabla\rho)^2 \quad . \quad . \quad . \quad (26)$$

and the first term is dropped when the Thomas–Fermi approximation is made. If the last term were absent, this approximation would lead to a sharp edge to the density distribution, and would certainly involve considerable error near the surface. In the present treatment however the presence of the last, potential surface energy, term already ensures finite density and potential gradients, so that the error involved in the approximation is smaller. Indeed the calculations made by Swiatecki (1951) indicate that the contributions of kinetic energy to the total surface energy are given quite well by the Thomas–Fermi approximation for surface thicknesses of 2×10^{-12} cm or more (see fig. 2). The validity of the approximation will be discussed further in § 7.

§ 5. Surface Thickness and Energy

With the above assumptions the energy density is given by

$$H = \rho E_0\{(1-\rho/\rho_0)^2 - 1\} + \tfrac{1}{2}B(\nabla\rho)^2. \quad . \quad . \quad . \quad . \quad . \quad (27)$$

It is now convenient to introduce the dimensionless variable

$$y = \rho/\rho_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and the length b defined by

$$b^2 = \tfrac{1}{2}B\rho_0/E_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

so that

$$H = \rho_0 E_0\{y^3 - 2y^2 + b^2(\nabla y)^2\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

The total energy must be minimized subject to the normalizing condition

$$\int y d^3x = (4\pi/3)r_0^3 \,.\, A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

(for a nucleus of a large number A of nucleons), and this leads to the differential equation for the density distribution

$$2b^2\nabla^2 y = d/dy(y^3 - 2y^2 + \lambda y) \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

where λ is the Lagrange multiplier to be determined by the condition (31).

It is evident that in a large nucleus the left side of (32) must be very small at the centre, where also $y \simeq 1$, so that $\lambda \simeq 1$; and the error made by putting $\lambda = 1$ is in fact of the relative order $A^{-2/3}$. Again for consideration of surface effects ∇^2 may be replaced by d^2/dx^2, where x is the distance

measured inwards from the surface; these assumptions together are equivalent to the discussion of a semi-infinite slab as made by Swiatecki. This gives

$$b^2(dy/dx)^2=y(1-y)^2$$
$$y=(\tanh x/2b)^2=F(x). \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Then it will be approximately true for a large spherical nucleus of outer radius R that

$$y(r)=F(R-r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The total energy of the system is then

$$\int H d\tau=-AE_0+2\int \tfrac{1}{2}B(\nabla\rho)^2 d\tau \quad . \quad . \quad . \quad . \quad . \quad (35)$$

of which the second term represents the surface energy, equal to

$$2\rho_0 E_0 \int b^2(dy/dx)^2\,(4\pi r_0^2 A^{2/3})dx=E_0 A^{2/3}\,(8b/5r_0). \quad . \quad . \quad . \quad (36)$$

Agreement with the experimental value requires therefore that

$$b=(5/8)r_0=0{\cdot}75\times 10^{-13}\ \text{cm} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and with this value of b it happens that the surface thickness also agrees with experiment. A reasonable measure of the surface thickness is provided by the value of $\int y(1-y)\,dx$; and if comparison is made on this basis between the distribution (33) and the often used form $y=[1+\exp(-x/a)]^{-1}$, the corresponding value of a should be $2b/3$, i.e. $0{\cdot}5\times 10^{-13}$ cm, in good agreement with the evidence from electron scattering.

It should be emphasized that this excellent agreement is rather fortuitous and linked with the assumption of the simple form (24) for $E(\rho)$; reasonable changes in the latter might alter the values of surface energy and thickness by some 20%; it would then however be difficult to choose a value of b to obtain agreement for both, and the above result therefore suggests that the value of the compressibility coefficient implied by (24) is near to the truth.

Finally the outer radius R determined by (31) and (34) is found to be

$$R=r_0A^{1/3}+2b=(1{\cdot}2A^{1/3}+1{\cdot}5)\times 10^{-13}\ \text{cm}. \quad . \quad . \quad . \quad (38)$$

The r.m.s. radius has the pure $A^{1/3}$ dependence.

§ 6. The Potential

It is interesting now to examine the mean potential U, given by eqn. (18), of which an estimate has been made by the Thomas–Fermi approximation. When the energy density is written in the form (26), with constant B, eqn. (18) should be replaced by

$$\frac{d}{d\rho}\rho E(\rho)-B\nabla^2\rho-\frac{6{\cdot}0}{2M^*}\rho^{2/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

and for the actual density distribution satisfying eqn. (32) this becomes

$$U=-E_0-\{6{\cdot}0/(2M^*)\}\rho^{2/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

inside the nucleus, and of course $U=0$ outside. A noteworthy feature of

this potential is the sharp edge at $r=R$, where the potential well is immediately 15 MeV deep. This result is independent of the detailed assumptions about the form of $E(\rho)$. Taken at its face value this would remove a difficulty, the apparent extension of the potential field significantly beyond the density distribution, that has been discussed by Drell (1955). This conclusion however may be rather sensitive to the use of the Thomas–Fermi approximation, and also to the representation of the pseudopotential in the form (12).

If the relation (16) is also employed, involving the additional assumption that scattering in even states is dominant, then the choice (37) implies.

$$M/M^*=1+0{\cdot}5y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and the potential (40) is completely determined ; the central depth of the well is $V_0=67$ MeV. This potential and the density distribution are shown in fig. 1.

Consider now the effect of this potential on an incident nucleon. Leaving aside for a moment the finite surface thickness, the particle will satisfy the wave equation

$$\{-(1/2M)\nabla^2-E\}\Psi=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

outside the nucleus, and inside

$$\{-(1/2M^*)\nabla^2-V_0-E\}\Psi=0 \quad . \quad . \quad . \quad . \quad (43)$$

which may also be written as

$$\{-(1/2M)\nabla^2-(M^*/M)V_0+E(1-M^*/M)-E\}\Psi=0 \quad . \quad (44)$$

so that the effective potential acting on the nucleon, when its inertia is supposed constant, is

$$V_{\text{eff}}=-(M^*/M)V_0+E(1-M^*/M) \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and with the corresponding numerical value of $M^*/M=0{\cdot}67$.

$$V_{\text{eff}}=-(45-0{\cdot}33E) \text{ MeV} \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

which is in good agreement with the experimental data at low and moderate energies (Adair 1954, Feshbach *et al.* 1954, Melkanoff *et al.* 1956).

In speaking of this effective potential it must also be remembered that the boundary conditions at the nuclear surface are changed. The proper conditions are that Ψ is continuous and that

$$(1/M)\,(d\Psi/dr)_{\text{ext}}=(1/M^*)\,(d\Psi/dr)_{\text{int}} ;$$

this discontinuity in gradient has the effect of strengthening the wave reflected from the surface. When the finite thickness of the surface is taken into account this means that the edge of the potential well appears a little steeper than it actually is, as may also be seen from the following analysis. In the neighbourhood of the surface the wave equation may be approximated in the one-dimensional form

$$\{-d/dx\,(1/2M^*)\,d/dx+U(x)-E\}\Psi=0 \quad . \quad . \quad . \quad . \quad (47)$$

and this may be transformed into a Schrödinger equation with constant mass by the transformations

$$dz=f^{1}dx, \chi=f^{1/2}$$

$$\text{with } f^2=(M/M^*)=1+0{\cdot}5\,y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The resultant wave equation for χ as a function of Z is

$$\left[-\frac{1}{2m}\frac{d^2}{dz^2}+\left\{U(x)+\frac{1}{4m}\left(f^{1/2}\frac{d}{dx}f^{1/2}\frac{df}{dx}\right)\right\}x=x(z)-E\right]\chi=0. \quad (49)$$

The potential U_{eff} occurring in this equation is shown in fig. 2 as a function of z in comparison with $U(x)$; the main effect is a foreshortening one due to the change from x to z.

This phenomenon would be observed as an apparent difference between measurements of the surface thickness obtained from electron scattering and from the naive interpretation of nucleon scattering. The effect expected is of the order of 10% ; experimentally, the thicknesses deduced from the interpretation of proton scattering experiments appear slightly smaller than those deduced from electron scattering (Melkanoff *et al.* 1956).

§ 7. Consistency

The potential $U(x)$ has been estimated by the Thomas–Fermi approximation, and it should now be asked whether the actual density distribution corresponding to the use of this potential would reproduce the density shown in fig. 1.

It can be seen from fig. 2 that the variation of M^* has a rather small effect on the shape of the potential. The topmost occupied level has an energy $-E_0$, so that the depth of the potential, measured from this level, is equivalent, very nearly, to the potential $(M^*/M)\{E_0+U(x)\}$ for a particle of fixed mass M ; this potential is also shown in fig. 2.

According to eqn. (40),

$$-2M^*(E_0+U)=(6{\cdot}0)\rho^{2/3} \quad . \quad . \quad . \quad . \quad . \quad (50)$$

which is another expression of the Thomas–Fermi approximation that has been made. Swiatecki (1955 a) has suggested an improved form of this relation between density and potential depth, derived from consideration of the density in a potential of uniform slope. For the present problem, with the effective potential given by eqn. (50), this relation (eqn. (6)) takes the form

$$\rho/\rho_0=(2\pi\sqrt{3})^{-1}\,(r_0/l)^3\phi\{1{\cdot}03\,(l/r_0)^2y^{2/3}\} \quad . \quad . \quad . \quad . \quad (51)$$

where $\phi(\xi)$ is a function given numerically by Swiatecki, and l is a measure of the inverse slope of the potential ; in this case

$$(r_0/l)^3=1{\cdot}03\,r_0\,(dy^{2/3}/dx)=1{\cdot}10\,y^{1/6}\,(1-y) \quad . \quad . \quad . \quad . \quad (52)$$

from eqns. (33) and (37).

Fig. 1

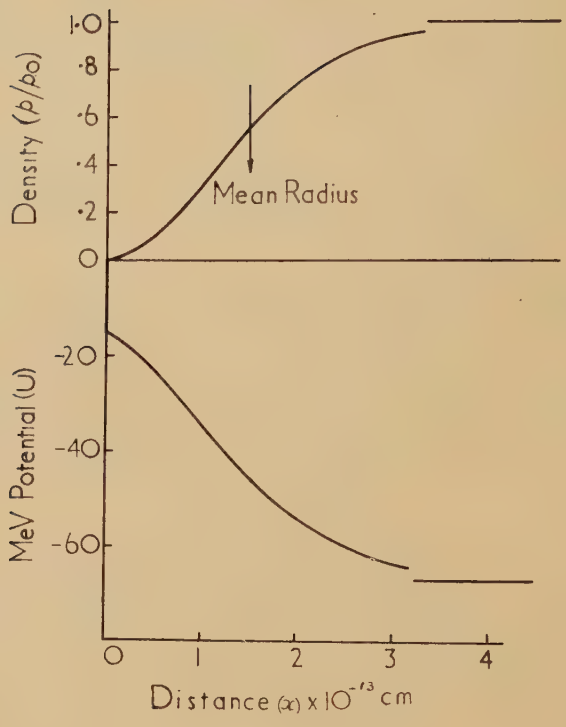
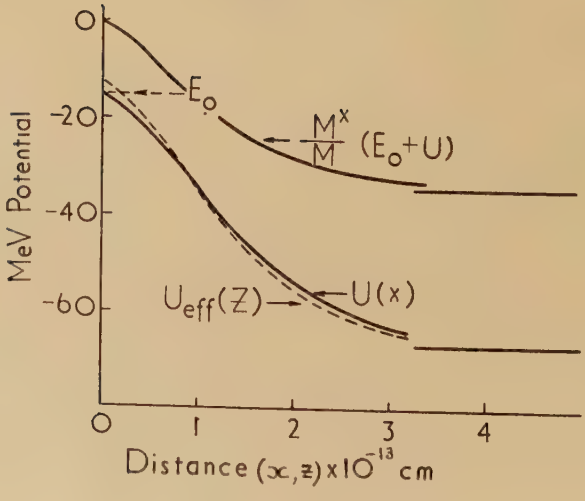
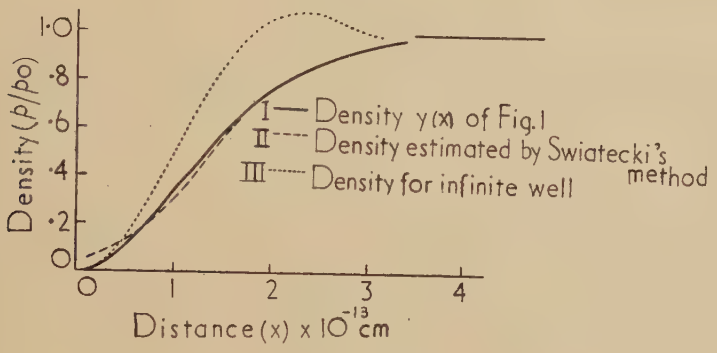

Density (ρ/ρ0)
1·0
·8
·6
·4
·2
0
Mean Radius
MeV Potential (U)
-20
-40
-60
0
1
2
3
4
Distance (x) x 10^-13 cm

Fig. 2


MeV Potential
0
-20
-40
-60
E0
M^x/M (E0+U)
Ueff(Z)
U(x)
0
1
2
3
4
Distance (x,z) x 10^-13 cm

For large l, eqn. (51) reproduces $\rho/\rho_0=y$. The density calculated with l given by eqn. (52) is shown in fig. 3 and compared with the original distribution $y(x)$. The difference is quite small, and the jump in the potential U at $x=0$ can be expected further to reduce the discrepancy. The third curve in fig. 3 shows the density distribution corresponding to a square well of infinite depth with its edge at $x=0$, as calculated by Swiatecki (1951).

Fig. 3



§ 8. Summary

The following assumptions have been made in this simplified model :

(i) The general assumptions of Brueckner's self-consistent field model.

(ii) The description of the two-particle scatterings in the finite system. by a point interaction derived from consideration of the infinite medium.

(iii) The retention only of quadratic terms in the velocity dependence of the potential, and that the coefficient t_1 is constant.

(iv) The replacement of the volume energy term by the approximation (24).

(v) The use of the Thomas–Fermi approximation.

Assumptions (i) and (ii) are the essential hypotheses upon which the present model is based, and no attempt has been made to relate the ' pseudopotential ' used here to any actual potential. Assumptions (iii) and (iv) are made primarily to simplify the algebra. Assumption (v) is made to obtain a simple picture of the form of the solutions of the self consistent field eqns. (17) and (18) ; the arguments § 7 suggest that it is not seriously in error.

There are three remaining parameters in the ' pseudopotential ' that is employed ; two of these E_0 and ρ_0 have been chosen to accord with the experimental values of the binding energy per particle and with the observed nuclear radius. The remaining parameter B and b can then be chosen to give both the correct surface energy and surface thickness. (In the discussion of the surface thickness the distinction between the model and true wave functions has been ignored.)

With this simple form it is also found that the potential well has a depth of 15 MeV at a distance $1{\cdot}5 \times 10^{-13}$ cm beyond the mean radius; this result may be sensitive to assumption (ii). The effective central well-depth for nucleons of incident energy E is $(45 - 0{\cdot}33\,E)$ MeV, in fair agreement with observation. The model also suggests that the potential should appear to have a slightly thinner surface for incident nucleons than would be expected from the measurements of density distribution by electron scattering.

Acknowledgments

The author is indebted to Professor R. E. Peierls, Dr. B. H. Flowers, Dr. A. M. Lane and Dr. J. Bell for a number of valuable discussions.

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CVIII. *The Nuclear Spin–Orbit Coupling*

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Abstract

Analysis of the nucleon–nucleon scattering around 100 mev has determined the spin–orbit coupling part of the two-body scattering matrix at that energy, and a reasonable extrapolation to lower energies is possible. This scattering amplitude has been used, in the spirit of Brueckner's nuclear model, to estimate the resultant single-body spin–orbit coupling for a single nucleon interacting with a large nucleus. This resultant potential has a radial dependence approximately proportional to $r^{-1}\, d\rho/dr$, and with a magnitude in good agreement with that required to explain the doublet splittings in nuclei and the polarization of nucleons scattered elastically off nuclei.

§ 1. Introduction

The coupling between the spin of a single nucleon and its orbit within a nucleus is manifested by the large splitting between the levels of a doublet in the nuclear excitation spectrum, interpreted in terms of a shell model, and by the strong polarization detected in the scattering of nucleons by nuclei. It is well known (Sternheimer 1955 and references given therein) that both these effects can be described approximately by an attractive potential $(\sigma \,.\, l)U(r)$ between a single nucleon and the whole nucleus, and that the strength of this potential is some twenty times greater than the relativistic Thomas term associated with the mean central potential.

Attempts have been made to explain this large value of the interaction as an amplification due to mesonic effects; in terms of radiative corrections (Chisholm and Touschek 1953) or in terms of a reduced inertial mass of the nucleon (Johnson and Teller 1955, Skyrme 1955). Our present knowledge of the mesonic fields within a nucleus is so uncertain, however, that an analysis along these lines remains speculative. A simpler, and not necessarily inconsistent, approach lies in an analysis in terms of two-body interactions. The only type of interaction which gives a spin–orbit coupling in the first approximation is the two-body spin–orbit force (Hughes and Le Couteur 1950, Elliott and Lane 1954, Blin-Stoyle 1955). The introduction of this force is however an additional assumption, and attempts have been made to explain the spin–orbit coupling in terms of higher order effects of the two-body tensor force,

† Communicated by Dr. B. H. Flowers.

which latter must also be present ; the calculations of Keilson (1951) do not suggest however that this is a good explanation, but the evaluation of higher order effects is a very complicated matter.

In the model of nuclear matter that has been discussed by Brueckner (1954) and co-workers in a series of papers, the two-body interactions are introduced through the two-body scattering matrix. From their original viewpoint the scattering matrix was to be determined by the experimental information on nucleon–nucleon scattering ; subsequently it was seen, that it was necessary to recalculate the scattering matrix for two particles interacting in the presence of the mean field of the other nucleons and subject to the exclusion principle. Apart from the complication introduced by this extra calculation there is the additional uncertainty due to ignorance of the potentials which give rise to the scattering and of how they may themselves be modified within nuclear matter. In this paper we have adopted the original point of view with regard to the relevant part of the scattering matrix.

An approximate determination of the scattering matrix has recently become possible as a result of experimental and theoretical developments. The angular distribution of the n–p polarization at 95 MeV has been measured by Hillman and Stafford (1956), and this has made possible a phase-shift analysis at that energy by Phillips (private communication), these phase-shifts are in close agreement with those given by Feshbach and Lomon (1956), whose results give also an estimate of the energy variation of the phases. The energies important in the present analysis lie below 100 MeV, and approximate analytical formulae have been determined to represent the spin–orbit part of the scattering amplitude over this energy range ; the uncertainties in this could be reduced by the extension to lower energies of the measurements of polarization.

This scattering amplitude has been interpreted for the finite nuclear system in a manner similar to that employed by Skyrme (1956) in his discussion of the nuclear surface ; as in that work the present discussion assumes a standard nucleus with equal numbers of protons and neutrons and does not allow for a possible difference between their distributions. Our analysis is closely related to that for a two-body spin–orbit potential, and may in some respects be regarded as an extension of the work of Blin-Stoyle (1955) ; the relation between these two approaches is discussed below.

§ 2. The Scattering Amplitude

The amplitude f that describes the scattering of two free nucleons is a scalar function of $\mathbf{k}$, $\mathbf{k}'$, the initial and final relative momenta of the nucleons, and of their spins $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. It is unnecessary to introduce the isotopic spin, provided that the initial and final states are properly antisymmetrized, because the p–p amplitude is part of the n–p one. The possible forms of spin-dependence of f have often been given, e.g. by Stapp (1955) ; those parts independent of spin are obviously irrelevant

to our purpose ; terms that are bilinear in the two spins also give no contribution to the resultant single-particle spin–orbit coupling, because the direct integrals vanish after averaging over the spins of one nucleon and the exchange integrals only give spin independent contributions after averaging. The only term that remains is that one linear in the spins, of the form

$$A=i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\,.\,(\mathbf{k}'\times\mathbf{k})F(\mathbf{k}^2,\,\mathbf{k}'^2,\,\mathbf{k}\,.\,\mathbf{k}') \quad . \quad . \quad . \quad (1)$$

where F is a scalar function of its arguments : on the energy shell $\mathbf{k}^2=\mathbf{k}'^2$.

For the comparison with the analysis of the experimental information the amplitude (1) must be related to the phase shifts. In terms of the scattering angle θ, and with the notation of Stapp (*loc. cit.*, eqn. (10) and table B)

$$k^2\sin\theta F(k^2,\,k^2,\,k^2\cos\theta)=-\tfrac{1}{4}\sqrt{2}(M_{10}-M_{01}). \quad . \quad . \quad . \quad (2)$$

The matrix elements M are expressible as series of harmonics with coefficients depending upon the phase shifts ; if only those terms are retained that appear to be important over the energy range considered,

$$\begin{aligned} M_{01}=&(1/4ik)(\sin\theta/\sqrt{2})[3R_{11}{}^1-3R_{11}{}^2-\sqrt{6}R^2\\ &+\cos\theta(3R_{21}{}^1+5R_{21}{}^2-8R_{21}{}^3+3\sqrt{2}R^1-4\sqrt{3}R^3)]\\ M_{10}=&(1/4ik)(\sin\theta/\sqrt{2})[-2R_{11}{}^0+2R_{11}{}^2-\sqrt{6}R^2 \quad . \quad (3)\\ &+\cos\theta(-6R_{21}{}^1+6R_{21}{}^3+3\sqrt{2}R^1-4\sqrt{3}R^3) \end{aligned}$$

where the R coefficients are functions of the phase shifts δ and admixture parameters ϵ (arising from the tensor interactions) given by Stapp (eqn. (39)). In the present problem it is not however really the scattering amplitude that is wanted, but the ' reactance amplitude ' corresponding to a scattered wave orthogonal to the incident wave ; this is obtained from the scattering amplitude by replacing $[\exp(2i\delta)-1]$ by $[2i\tan\delta]$. Then, for $L=J$,

$$R^J_{J,\,1}=2i\tan\delta^J_J \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and for the mixed triplet states with $L=J\pm1$,

$$\begin{aligned} R^J_{J+1,\,1}&=2i(\cos^2\epsilon^J\tan\delta^J_{J+1}+\sin^2\epsilon^J\tan\delta^J_{J+1})\\ R^J&=i\sin^2\epsilon^J(\tan\delta^J_{J+1}-\tan\delta^J_{J-1}). \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

(The quantities R^J do not appear in the expression (2).)

We want now to represent A as a simple analytic function of the momenta $\mathbf{k}$ and $\mathbf{k}'$. Comparison of (1), (2) and (3) suggests that we should take F as a linear function of $\mathbf{k}\,.\,\mathbf{k}'$ at a given energy,

$$F=A(\mathbf{k}^2,\,\mathbf{k}'^2)+C(\mathbf{k}^2,\,\mathbf{k}'^2)\mathbf{k}\,.\,\mathbf{k}'. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In principle the values of A and C on the energy shell, when $\mathbf{k}^2=\mathbf{k}'^2$, can be found from the experimental phase shifts (by using formulae (2) to (5) above), but at present information sufficient to determine the phase shifts is available at only one energy, around 95 Mev, in the important range, so that this must be supplemented by semi-empirical estimates of

the energy-dependence of the phase shifts ; for this we have used the work of Feshbach and Lomon (1956).

The phase shifts that occur in R can be expanded in a series of odd powers of k, so in principle A and C in eqn. (6) could be expanded, on the energy shell, in powers of k^2. As a rough approximation A can be represented by a linear expression, and this will naturally be generalized to points off the energy shell, using the condition of hermiticity, by assuming

$$A(k^2, k'^2)=A_0-\tfrac{1}{2}(A_1k^2+A_1{}^*k'^2). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The energy variation of C cannot be represented properly in this way, owing to the behaviour of the ^3S phase-shift $\delta_0{}^1$; this phase goes to π at zero energy and must pass through $\frac{1}{2}\pi$ somewhere around 20 MEV, at which point C will have a pole. Such low energies do not however make a significant contribution to the spin–orbit coupling so we look for a rough approximation to C over the range 40–100 MEV. In our subsequent analysis we shall first make the crude approximation

$$C(k^2, k'^2)=\text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

which is very poor, but sufficient as a guide to the expected effects of the C term.

It will appear that in fact the most important part of F is the ' forward-scattering ' part, and it is convenient to include in this part the contributions that will arise from exchange. Then, in a ' standard ' nucleus with four nucleons in each orbital state, the amplitude concerned is

$$\begin{aligned} F_0&=F(\theta=0)+\tfrac{1}{2}F(\theta=\pi) \\ &=(3/2)A(k^2, k^2)+(1/2)C(k^2, k^2)k^2 \quad . \quad . \quad . \quad . \quad (9) \end{aligned}$$

where the factor $\frac{1}{2}$ is the expectation value of the charge exchange operator P_τ. It is possible to handle somewhat more complicated approximations to F_0 ; in particular a quadratic polynomial in k^2 seems to be a fair enough approximation over the important energy range, and the inadequacy of experimental or theoretical knowledge of the correct value of F does not justify more precise calculations at present. Such an approximation is used for our numerical estimates in § 4.

§ 3. Nuclear Calculation

The method of calculation is similar to that employed by Skyrme (1956) ; as in eqn. (12) of that reference a pseudopotential is introduced defined by

$$V_{12}=(-4\pi h^2/M)A(\mathbf{k}, \mathbf{k}')\delta(\mathbf{x}_1-\mathbf{x}_2) \quad . \quad . \quad . \quad . \quad (10)$$

where A is the reactance amplitude, and the normalization is so chosen as to reproduce the scattering amplitude in Born approximation. In a matrix element of V_{12}, $\mathbf{k}$ and $\mathbf{k}'$ are to be interpreted as differential

operators acting upon the states to the right and to the left respectively,

$$\left.\begin{aligned}\mathbf{k}=\tfrac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)&=-\tfrac{1}{2}i(\nabla_1-\nabla_2), \quad \text{on the right}\\ \mathbf{k}'=\tfrac{1}{2}(\mathbf{k}_1'-\mathbf{k}_2')&=\tfrac{1}{2}i(\nabla_1-\nabla_2), \quad \text{on the left}\end{aligned}\right\} \quad . \quad . \quad (11)$$

In a matrix element integrated over all space, integration by parts leads to a relation between these operators, analogous to conservation of momentum,

$$\mathbf{k}_1'+\mathbf{k}_2'=\mathbf{k}_1+\mathbf{k}_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In the spirit of Brueckner's model of the nucleus, the contribution of the spin–orbit interaction to the total energy of the system is found from the expectation value of $\sum_{i<j} V_{ij}$ for a determinantal wave-function of single-particle states $\phi_i(\mathbf{x})$. The terms that involve a particular $\phi_0(\mathbf{x})$ are

$$\textstyle\sum_i \iint \phi_0^*(\mathbf{x}_1)\phi_i^*(\mathbf{x}_2)V_{12}[\phi_0(\mathbf{x}_1)\phi_i(\mathbf{x}_2)-\phi_0(\mathbf{x}_2)\phi_i(\mathbf{x}_1)]\,d\mathbf{x}_1\,d\mathbf{x}_2. \quad . \quad (13)$$

and the summation over states i may include the state ϕ_0 without error because of the antisymmetry.

When the amplitude A is expressed as a polynomial in $\mathbf{k}$ and $\mathbf{k}'$, the direct integrals in (13) will lead to sums such as $\sum|\phi_i(\mathbf{x})|^2$, $\sum\phi_i^*(\mathbf{x})\nabla\phi_i(\mathbf{x})$, etc. ; those that occur in our subsequent analysis can be expressed in terms of

$$\left.\begin{aligned}\rho&=\textstyle\sum|\phi_i(x)|^2\\ \tau_{\mu\nu}&=\textstyle\sum(d\phi_i^*/dx_\mu)(d\phi_i/dx_\nu), \qquad \tau=\tau_{\mu\mu}\\ \xi&=\textstyle\sum(\nabla^2\phi_i^*)(\nabla^2\phi_i)\end{aligned}\right\} \quad . \quad . \quad . \quad (14)$$

The exchange integrals involve similar sums that are outer products in spin and isotopic spin space, instead of the inner products that occur in (14). With the assumption of a standard nucleus with four particles in each state, this outer product will be a unit operator I in the spin spaces, so that

$$\textstyle\sum\phi_i(\mathbf{x})\phi_i^*(\mathbf{x}')=\frac{1}{4}I\sum\phi_i^*(\mathbf{x}')\phi_i(\mathbf{x}). \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Finally in this perturbation calculation all these sums may be evaluated for the uncoupled states, and provided that the spin states are equally occupied these may be taken to be real in space, so that the conjugation sign may be dropped in the above definitions.

As a first step the energy (13) has been evaluated with the assumptions (7) and (8) ; we define the constants a, b and c so that

$$(-4\pi\hbar^2/M)A=i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\,.\,(\mathbf{k}'\times\mathbf{k})[a-\tfrac{1}{2}(b\mathbf{k}^2+b^*\mathbf{k}'^2)+c\mathbf{k}\,.\,\mathbf{k}']. \quad . \quad (16)$$

The calculation is straightforward ; the use of the relation (15) enables the exchange integrals to be expressed as multiples of the direct ones : for the a and b terms this multiple is $\frac{1}{2}$, for the c term it is $-\frac{1}{2}$. The result may be expressed in the form

$$\textstyle\int\phi_0^*(\mathbf{x})V(\mathbf{x})\phi_0(\mathbf{x})\,d\mathbf{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where then $V(x)$ is the resultant single-particle potential. Provided that the sums (14) have spherical symmetry, i.e.

$$\begin{aligned}\rho(\mathbf{x})&=\rho(r)\\ \tau_{\mu\nu}(x)&=(\tfrac{1}{3})\tau(r)\delta_{\mu\nu}+\tau_1(r)(x_\mu x_\nu-r^2\delta_{\mu\nu}/3), \quad . \quad . \quad . \quad (18)\end{aligned}$$

$V(x)$ can be expressed in the form $(\boldsymbol{\sigma} \cdot \mathbf{l})U(r)$. The expression for U can be simplified further if it is supposed that the radial part of ϕ_0 is real; this restriction essentially means that ϕ_0 describes a stationary state.

Then we obtain from (16),

$$\begin{aligned} U(r)=&(\tfrac{3}{4}a)r^{-1}d\rho/dr \\ &-(3/16)(b-c/3)[(5/3)r^{-1}d\tau/dr-(2/3)rd\tau_1/dr \\ &-(10/3)\tau_1-r^{-1}d\rho/dr(\nabla^2)] \\ &+(3/32)br^{-1}d(\nabla^2\rho)/dr \\ &+(3/32)(b+2c/3)[(rd/dr+5)(r^{-1}d/drr^{-1}d\rho/dr)] \quad . \quad . \quad (19) \end{aligned}$$

where b has been written for the real part of b, since only that occurs.

The last term can be simplified if only terms of order $1/r$ at the surface are retained; then the last two terms together give

$$(3/16)(b+c/3)r^{-1}d(\nabla^2\rho)/dr. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

These last terms involve second or higher derivatives of the density, so that their average through the surface vanishes (for sufficiently large r); and as might be expected their contribution to the spin–orbit coupling is rather small; it is also rather uncertain, depending sensitively upon the density distribution and upon the shape of the wave function ϕ_0. If these are neglected the remaining terms of U can be obtained in a simple way that can conveniently be extended to more complicated expressions for the amplitude A. This method depends upon the observations that the vector product $\mathbf{k}'\times\mathbf{k}$ is equivalent with $(\mathbf{k}_2-\mathbf{k}_2')\times\mathbf{k}$, on account of the relation (12) that may be used inside the integral (13), and that the factor $(\mathbf{k}_2-\mathbf{k}_2')$ is equivalent to an operation of differentiation upon the product $\phi_i^*(x)\phi_i(x)$. It is this factor which produces the first derivatives occurring in the principal part of U; the later terms containing higher derivatives arise from the difference between $\mathbf{k}_2$ and $\mathbf{k}_2'$ in the terms of F. Thus the retention only of the leading terms of U is equivalent with putting $\mathbf{k}_2=\mathbf{k}_2'$, and so $\mathbf{k}=\mathbf{k}'$, in F, after the inclusion of the exchange contributions.

This last prescription means that then only the 'forward' part F_0 of F is required as in eqn. (9). Then the evaluation of (13) involves first the averaging of $\mathbf{k}F_0(\mathbf{k}^2)$ over $\mathbf{k}_2$ at the position $\mathbf{x}$ ignoring the gradients of density etc., and secondly a differentiation with respect to position to take care of the factor $(\mathbf{k}_2-\mathbf{k}_2')$: the result can be expressed in terms of gradients of the quantities defined in eqn. (14). It can easily be verified that the assumption $F_0=(3/2)(a-bk^2)+\frac{1}{2}ck^2$ indeed reproduces the first two terms in the result (19) that we found above.

As was mentioned in the previous section a fair approximation can be made to F_0 in the form

$$(-4\pi\hbar^2/M)F_0=\alpha-\beta k^2+\gamma k^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and then this method can easily be applied to find $U(r)$. If the

non-diagonal terms such as τ_1 are omitted the result is

$$\begin{aligned} U(r)=&\tfrac{1}{2}\alpha r^{-1}d\rho/dr \\ &-(1/8)\beta[(5/3)r^{-1}d\tau/dr-r^{-1}d\rho/dr(\nabla^2)] \\ &+(1/32)\gamma[(7/3)r^{-1}d\xi/dr-(14/3)r^{-1}d\tau/dr(\nabla^2) \\ &+r^{-1}d\rho/dr(\nabla^4)]. \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

As written this expression is not properly hermitian but this does not cause any difficulty in the following applications where ∇^2 is equivalent to a real operator.

The evaluation of these formulae requires a knowledge not only of the density distribution, but also of the distribution of τ etc. For this we have employed the Thomas–Fermi approximation, as in Skyrme (1956), according to which

$$\tau=(3/5)(6{\cdot}0)\rho^{5/3}\,; \qquad \xi=(3/7)(36{\cdot}0)\rho^{7/3}. \quad . \quad . \quad . \quad (23)$$

Furthermore with the same approximation for a particle at the top of the Fermi distribution, energy $-E_0$,

$$-\nabla^2=(6{\cdot}0)\rho^{2/3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

so that for a particle at energy level E

$$\begin{aligned} -\nabla^2&=(6{\cdot}0)\rho^{2/3}+k^2 \\ k^2&=(2M/\hbar^2)(E+E_0). \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

With the help of (23) and (25), $U(r)$ defined by eqn. (22) can be written in the form

$$\begin{aligned} U(r)=&\tfrac{1}{2}r^{-1}(d\rho/dr)\{\alpha-\beta(4{\cdot}0\rho^{2/3}+\tfrac{1}{4}k^2) \\ &+\gamma[18{\cdot}0\rho^{4/3}+(10{\cdot}0\rho^{2/3})(\tfrac{1}{4}k^2)+(\tfrac{1}{4}k^2)^2]\}. \end{aligned} \quad . \quad . \quad . \quad (26)$$

Compared with eqn. (19) this neglects among other terms, those in τ_1 (defined in eqn. (18)). An estimate of the error made can be obtained by considering the momentum distribution in a Fermi gas bounded by an infinite plane wall, a model suggested by Swiatecki (1951). The anisotropy of the momentum distribution is most marked actually at the wall, where the density goes to zero ; at this point the effect would be to replace the factor (5/3) that multiplies $r^{-1}d\tau/dr$ in the second term of (22) by 1·4, and correspondingly in eqn. (26) the term $4{\cdot}0\rho^{2/3}$ that multiplies β would become $3{\cdot}6\rho^{2/3}$. Since this is only a 10% effect at its most important it is a small matter compared with other errors that can arise from our approximations.

§ 4. The Doublet Splitting

We consider in this section the doublet splitting for a nucleon at the top of the Fermi distribution, for which, in eqn. (25), $k^2=0$. Then the formula (26) gives

$$U_0(r)=\tfrac{1}{2}r^{-1}d\rho/dr[\alpha-(1{\cdot}1y^2/3)\beta+(1{\cdot}3y^{4/3})\gamma] \quad . \quad . \quad . \quad (27)$$

where we have put

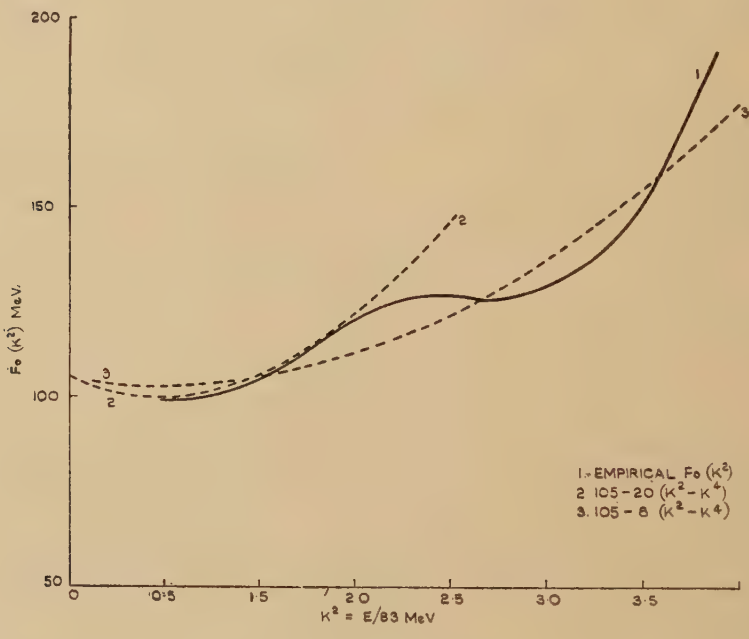
$$\rho=y\rho_0\,; \qquad \rho_0=(4\pi r_0^3/3)^{-1}=0{\cdot}138 \quad . \quad . \quad . \quad . \quad (28)$$

with $r_0=1{\cdot}2$, and the unit of length equal to 10^{-13} cm.

If the form (21) is assumed to be a good approximation the result (27) shows by comparison that the important range of $\mathbf{k}^2$ is from 0 to 1·1, corresponding to laboratory energies of 0 to 90 Mev, with the greatest contribution near $y=\frac{1}{2}$, around 55 Mev. A more physical argument is that the important values of $\mathbf{k}^2$ are those around the average of $\frac{1}{4}(\mathbf{k}_1-\mathbf{k}_2)^2$, where $\mathbf{k}_1$ is the momentum of the particle at the top of the Fermi distribution and $\mathbf{k}_2$ of another with which it interacts ; this average has the value $(2/5)k_F^2=0{\cdot}63$, with $r_0=1{\cdot}2$, giving an energy of 52 Mev.

Unfortunately polarization experiments have not yet been done at this low energy, so an extrapolation downwards has been needed of the

Fig. 1



The function F_0.

phase-shift analysis available at 95 Mev (Phillips, private communication). This has been made following the estimates given by Feshbach and Lomon (1956), which are in agreement with the phases determined at 95 Mev and higher energies. The function $F_0(k^2)$, defined by eqn. (9), determined empirically in this way is shown in fig. 1. An approximation to this curve for the lower energies has been taken in the form (21) with the coefficients

$$\alpha=105, \qquad \beta=\gamma=20 \text{ Mev} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

with 10^{-13} cm as the unit of length ; this approximation is also shown in the figure.

With the values given by (29), $U_0(r)$ becomes

$$U_0(r)=r^{-1}d\rho/dr[50{\cdot}2+13(y^{2/3}-0{\cdot}42)^2]. \quad . \quad . \quad . \quad . \quad (30)$$

The second term is rather small, reflecting the small dependence of F_0 on k^2 in this range ; a simple average over y, from 0 to 1, gives the value 1·3. Since the estimation of its expectation value is rather uncertain anyway, we shall adopt the simple formula

$$U_0(r)=51\cdot5r^{-1}d\rho/dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

With the perturbing potential $\boldsymbol{\sigma}\,.\,\mathbf{l}U(r)$, the splitting between levels with $j=l\pm\frac{1}{2}$ is

$$\Delta E=-(2l+1)\langle U(r)\rangle. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

For a light nucleus it may not be unreasonable to assume a parabolic density distribution

$$\rho(r)=\rho_1(1-r^2/R^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

for which, independently of the state of the nucleon considered, (31) and (32) give

$$\Delta E=(2l+1)(51\cdot5)(2\rho_1/R^2). \quad . \quad . \quad . \quad . \quad . \quad (34)$$

If we write the standard radius $r_0A^{1/3}$ as R_0, the proper normalization of the density (33) gives the condition

$$\rho_1/\rho_0=2\cdot5(R/_0R)^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

so that ρ_1 may be eliminated from (34) to give

$$\Delta E=24\cdot8(R_0/R)^5(2l+1)A^{-2/3}. \quad . \quad . \quad . \quad . \quad . \quad (36)$$

If now R be chosen so that the mean square radius is equal to that of a sphere of radius R_0, $(R/R_0)=1\cdot18$ and $\Delta E=10\cdot7\ (2l+1)A^{-2/3}$. This gives for ^{15}N and ^{17}O splittings equal to 5·3 and 8·1 Mev respectively, to be compared with the experimental values of 6·3 and 5·1 Mev respectively.

Another estimate of the density distribution in light nuclei may be found from a shell model with oscillator wave functions : the calculation of the expectation value in (32) is then equivalent to the calculation made by Lane and Elliott (1954) in the limit of zero range, as will be discussed below. The calculated splittings depend upon the scale constant r_1 of the wave functions (supposed to contain the exponential factor $\exp(-\frac{1}{2}r^2/r_1^2)$), and are for

$$^{15}\mathrm{N}:\ 23\cdot9(r_0/r_1)^5\ ; \qquad ^{17}\mathrm{O}:\ 33\cdot0(r_0/r_1)^5. \quad . \quad . \quad . \quad . \quad (37)$$

For the same value of r_1 the ratio of these is close to the ratio found for a parabolic distribution. To reproduce the observed splittings it would be necessary to assume

$$^{15}\mathrm{N}:\ r_1=1\cdot56\ ; \qquad ^{17}\mathrm{O}:\ r_1=1\cdot74. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

As has been pointed out by Lane and Elliott (1954) such a jump in the value of r_1 going across the ^{16}O closed shell appears to be necessary to explain the Coulomb energy differences of mirror nuclei ; their unpublished calculation gives the values 1·67 and 1·90 for the two nuclei, with a ratio similar to that required in (38). The equation of the mean square radius with $(3/5)R_0^2$ for ^{16}O gives $r_1=1\cdot56$.

Since the spin–orbit splitting is such a sensitive function of the density distribution, exemplified by the fifth power of r_1 that occurs in (37), and since it is a rather different measure of it from the Coulomb energy difference or the mean square radius, it is hard to say whether or no the formula (31) leads to results in accord with experiment. A tentative conclusion from the analysis given would be that the constant in that formula may be too small, possibly by as much as 30%.

In the region of heavy nuclei the experimental information near ^{208}Pb has been examined by Blin-Stoyle (1955), who employed the form (31) with an adjustable numerical constant. He found that the data could be well fitted by taking for the constant (K, in his notation) the value 57 (assuming that $r_0=1{\cdot}2$), which is 10% greater than our calculated value.

In noting these approximate agreements we should not overlook the assumptions made in deriving the simplified formula (26); these were first the 'forward scattering' approximation used to obtain (22) and secondly the use of the Thomas–Fermi approximation to express all quantities in terms of the density distribution. These approximations are very similar and both depend for their validity on the smallness of the density gradient. This condition is hardly satisfied at the nuclear surface, but once the forward-scattering approximation has been made the remaining error will be small because of the flatness of F_0 as a function of k^2; on the other hand the separate parts A and Ck^2 of F_0, as in eqn. (9), vary more rapidly with energy and the error in the forward scattering approximation may be appreciable. It will be shown in the § 5 how an alternative evaluation of (16) may be made which indicates, as might be expected, that the approximations have tended to underestimate the magnitude of the important momenta.

§ 5. Relation with Spin–Orbit Potential

The method that we have employed is essentially the same as that generally used in shell-model calculations, namely the calculation of the expectation value of a perturbing interaction in an unperturbed individual particle state; the difference is only that we have used a singular point-type interaction instead of a potential. The singularity arises just because we have chosen to use a finite polynomial expression for the scattering amplitudes; in so far as these polynomials can be regarded as approximations to the matrix elements of some potential, equivalent results should be obtained by working with that potential.

In our problem the amplitude A, eqn. (1), resembles closely the matrix element of the two-body spin–orbit potential, and we shall now enquire how far this correspondence can be carried. We shall assume a potential of the type

$$-(1-x+xP_\tau)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{r}_1-\mathbf{r}_2)\times(\mathbf{p}_1-\mathbf{p}_2)V(r_{12}) \quad . \quad . \quad . \quad (39)$$

such as has been considered by Lane and Elliott (1954) and other authors. In this expression P_τ is the charge exchange operator and x measures the

fraction of exchange potential. The matrix element of this potential between states of momentum $\mathbf{k}$ and $\mathbf{k}'$ for the n–p system is easily found to be

$$i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\,.\,(\mathbf{k}'\times\mathbf{k})[(1-x)W(|\,\mathbf{k}-\mathbf{k}'\,|)+xW(|\,\mathbf{k}+\mathbf{k}'\,|)] \quad . \quad (40)$$

where W is related to $V(r)$ by

$$W(q)=(8\pi/q^3)\int_0^\infty V(r)(\sin qr-qr\cos qr)r\,dr. \quad . \quad . \quad . \quad (41)$$

In particular for the form of V used by Lane and Elliott, the Yukawa form $V=V_0(\lambda/r)\exp(-r/\lambda)$, we obtain

$$W(q)=16\pi V_0\lambda^5(1+\lambda^2q^2)^{-2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

As compared with the general form of amplitude given by (1), the expression (40) derived from a static potential in Born approximation is of a special form, and we cannot expect to be able to express the observed scattering amplitude in this way. As we mentioned in the discussion following eqn. (6) the coefficient C in that equation has a singular behaviour at low energies, and this certainly cannot be reproduced from a potential. The physical meaning behind this is that ' final state interactions ' play an important role in the spin–orbit coupling between pairs of nucleons, and the use of a spin–orbit potential in Born approximation can only be regarded as a mathematical artifice. At higher energies where the phase shifts become small a direct physical interpretation as a potential might be more reasonable, but then the whole concept of a potential seems to be unsatisfactory for analysing nucleon–nucleon scattering.

It is possible however to use this idea of a potential to find the effect of using the form (19) for $U(r)$. This was derived from the form (16), which can be reproduced by expanding W to order q^2 with suitable values for the parameters V_0, λ, x. Thus to this order

$$\begin{aligned}&(1-x)W(|\,\mathbf{k}-\mathbf{k}'\,|)+W(|\,\mathbf{k}+\mathbf{k}'\,|)\\&\quad=16\pi V_0\lambda^5[1-2\lambda^2(\mathbf{k}^2+\mathbf{k}'^2)+4\lambda^2(1-2x)\mathbf{k}\,.\,\mathbf{k}']. \quad . \quad . \quad . \quad (43)\end{aligned}$$

The results of Lane and Elliott may also be expanded to relative order λ^2, giving then the splitting due to the amplitude F given by (43); their results are for

$$\begin{aligned}&{}^{15}\mathrm{N}: \quad \Delta E=(108/\sqrt{2\pi})(V_0\lambda^5/r_1{}^5)[1-[(70+40x)/9](\lambda^2/r_1{}^2)],\\&{}^{17}\mathrm{O}: \quad \Delta E=(150/\sqrt{2\pi})(V_0\lambda^5/r_1{}^5)[1-(10+3{\cdot}2x)(\lambda^2/r_1{}^2)]. \quad . \quad (44)\end{aligned}$$

In terms of the constants a, b and c of eqn. (16), determined by comparison with (43), the final factors in these expressions for the splittings are proportional respectively to

$$a-(b-0{\cdot}22c)(2{\cdot}5/r_1{}^2)\,; \qquad a-(b-0{\cdot}14c)(2{\cdot}9/r_1{}^2) \quad . \quad . \quad . \quad (45)$$

which are to be compared with the estimate

$$a-(b-0{\cdot}33c)(4{\cdot}0\rho^{2/3}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

derived from eqn. (26). The comparison suggests that the coefficient of b, and therefore the nuclear momenta, have been somewhat underestimated.

As we mentioned before the form (43), equivalent to (16), is not a realistic representation of the amplitude so the analysis in terms of a

potential is limited in application. Because the amplitude F_0 is a slowly varying function of energy the analysis in § 4 leads to substantially the same result as the use of a potential of zero range (when the exchange character is irrelevant). Such a potential implies that all the interaction between nucleons takes place in the ^{3}P state, but this is not really so and there is a considerable contribution from the ^{3}S–^{3}D coupled state which gives the C term in eqn. (6).

§ 6. Potential for Scattering

Another possible comparison of our results with experiment comes from the analysis of the polarization of nucleons elastically scattered from target nuclei. Our analysis has been based upon the possibility of using the form (6) for the amplitude (although this could easily be generalized); accordingly comparison should be limited to those energies for which it may be a fair approximation. In this section we shall refer to the calculations of Sternheimer (1955) analysing the scattering of 130 Mev protons from light nuclei; this energy is already rather high, so that we should be prepared for some inaccuracy in our hypotheses.

The values (29) do not give a good representation of F_0 over the larger energy range that is relevant for incident nucleons of 130 Mev; a better overall fit is obtained with the choice

$$\alpha=105, \qquad \beta=\gamma=8 \qquad . \; . \; . \; . \; . \; . \; . \quad (47)$$

which is illustrated in fig. 1.

In this case we must include the k^2 terms in the formula (26) for $U(r)$; k^2 was defined by eqn. (25), and in that equation the mass M should properly be the effective mass M^*. We adopt the values used by Skyrme (1956) for M^* and E_0, which give

$$\tfrac{1}{4}k^2=1{\cdot}75(1+0{\cdot}5y)^{-1} \; . \; . \; . \; . \; . \; . \; . \quad (48)$$

where as before y is the density in units of ρ_0; we also use the same density distribution, so that in the ' large nucleus ' approximation

$$-\tfrac{1}{2}r^{-1}d\rho/dr=0{\cdot}0767A^{-1/3}y^{1/2}(1-y)$$
$$y=\rho/\rho_0=[\tanh{(R-r/(1{\cdot}5))}]^2. \qquad . \; . \; . \quad (49)$$

The formula (26) thus gives

$$-A^{1/3}U(r)=K(1-y)y^{1/2} \; . \; . \; . \; . \; . \; . \; . \quad (50)$$

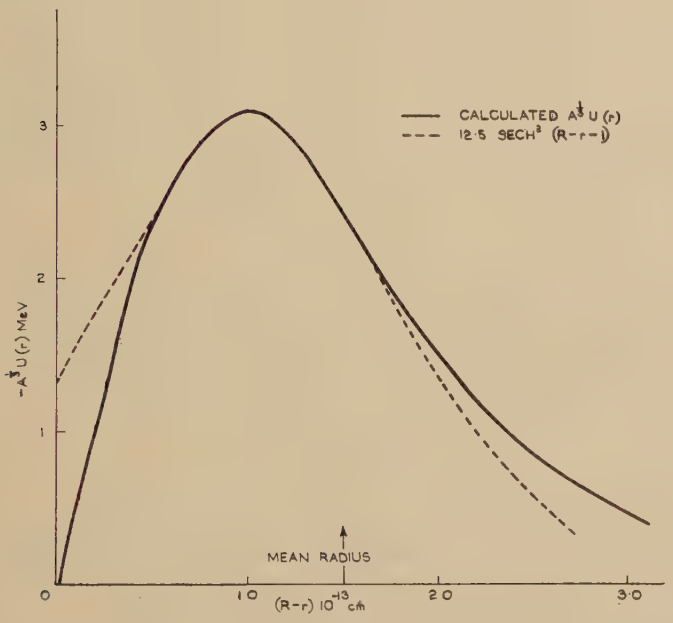
where the coefficient K varies from 8·85 to 10·25 as y goes from 0 to 1. For comparison we note that eqn. (31) gives $K=7{\cdot}9$, so that there is a 20% increase in the strength of the potentials compared with that felt by a particle at the top of the nuclear well.

Before we compare our potential with that used by Sternheimer, we should make allowance for the mass variation. In the approximate expression for the phase shifts used by Sternheimer the perturbing potential is multiplied by the factor (hk/T), where T is the kinetic energy of the nucleon and k its wave number. This factor is equal to $(2M^*/(E-V))^{1/2}$, where E is the energy of the incident nucleon and V

is the nuclear potential, and when the effective mass changes appreciably through the surface the potential V is increased so that the variation of this factor is not negligible. An effective potential has accordingly been estimated by multiplying (50) by this factor, expressed as a function of y with the same hypotheses. The resulting modified coefficient K' then varies from 8·4 to 6·8 ; this potential is shown in fig. 2 as a function of distance from the outer edge of the nucleus, and it is not significantly different from the potential (31).

The maximum value of the potential is about $3A^{-1/3}$ Mev and its width is very similar to that assumed by Sternheimer, whose potential is also shown in the figure, adjusted to have a coincident maximum ;

Fig. 2



The potential $U(r)$.

the maximum value is close to that found necessary by Sternheimer to explain the experimental results. The main difference is that the maximum of our potential occurs $0{\cdot}5\times10^{-13}$ cm outside the mean radius $r=r_0A^{1/3}$ instead of at that point (always speaking of the potential in a large nucleus, ignoring the r^{-1} factor outside). It is not possible to say without detailed calculation whether this and other small differences would help to remove the difficulties that Sternheimer found in fitting properly the angular distributions.

Summary

The general shape of the phase-shift analysis of the nucleon–nucleon scattering for energies up to 100 Mev now seems well established and it

has been possible to fix the spin–orbit part of the scattering amplitude with reasonable certainty.

This empirically determined amplitude has been applied in the spirit of Brueckner's model of the nucleus to find the effective single-particle spin–orbit coupling. The essential assumptions are that

(i) The energy of the nuclear system may be determined by using a 'model' wave function which is approximately that of a set of independent particles moving in a central scalar potential.

(ii) The energy change due to a perturbing interaction is approximately the sum of changes due to two-body interactions and can be evaluated by using the reactance amplitudes and model wave function.

(iii) The scattering associated with these two-body interactions is approximately the same as that between free nucleons, so far as the spin–orbit amplitude is concerned.

The numerical result that we obtain from our analysis appears to be in agreement with experimental evidence, within the limitations arising from ignorance of the nuclear wave functions. However this agreement really involves only one quantity, the strength of the coupling, so that it would be premature to claim that the method of calculation was thereby justified.

Acknowledgments

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CIX. CORRESPONDENCE

Stacking Faults in Cold-Worked Cobalt–Nickel Alloys

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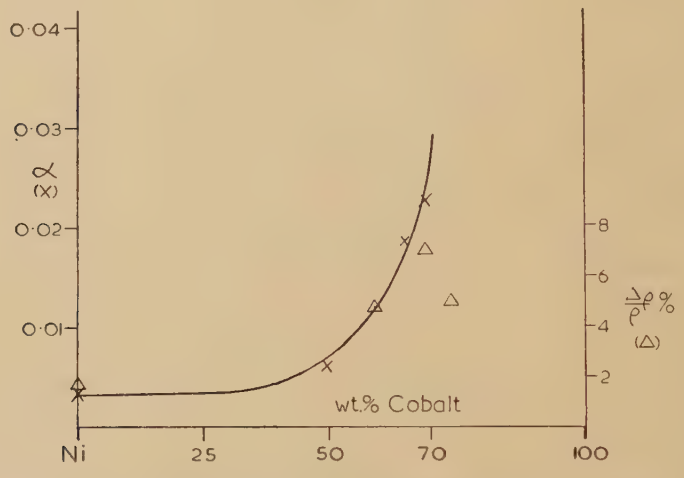
Warren and Warekois (1955) have shown that heavily cold-worked alpha-brasses contain deformation stacking faults in sufficient amounts to give measurable x-ray diffraction effects. The stacking fault parameter (Paterson 1952) increases with zinc content from $\alpha=0{\cdot}005$ at 10% zinc to $\alpha=0{\cdot}039$ at 35% zinc. In this note, we briefly record similar results we have obtained for cobalt–nickel alloys, and we discuss a possible correlation between the production of faults and the abnormally large increases in electrical resistivity on cold-working, and large work-hardening capacities, shown by certain alloys of critical composition.

Cobalt–nickel alloys were chosen because the free energy difference between the bulk face-centred cubic and hexagonal close packed forms changes sign in a certain composition range, thus indicating that the specific energy of a fault should be low. Electrical resistance measurements by Broom and Barrett (1953) have already suggested that stacking faults probably form in these alloys. Growth stacking faults ((1Δ) faults in Frank and Nicholas' 1953 notation) are found in considerable amounts in hexagonal cobalt naturally transformed from the cubic phase (Edwards and Lipson 1942, Wilson 1942), but recent work in this laboratory has shown that when the martensitic transformation is stress-aided, the faults are predominantly of the hexagonal deformation (2Δ) type. These (2Δ) faults are the hexagonal equivalent of the cubic deformation faults studied in the present work. Alloys containing from 50 to 69 weight-% cobalt were filed, passed through 250 mesh sieves and formed into diffractometer specimens in specially designed holders. Accurate line profile measurements were made by stepwise counting on a Norelco diffractometer, using cobalt radiation. The specimens were given various intermediate annealing treatments before final recrystallization at 800°C, and line profiles were measured at each stage. Values of the stacking fault parameter were obtained from the shifts in the positions of the $K\alpha_1$ peaks, which were located by the method described by Anantharaman and Christian (1953). Consistent values could be obtained from the movements of individual lines, as well as from the changes in separation, and the method is considered to be more reliable than that of measuring the centre of gravity of a line used by Warren and Warekois.

† Communicated by the Authors.

The stacking fault parameters for cobalt–nickel alloys are shown in fig. 1 ; the probable error is believed to be less than 10% when $\alpha \simeq 0{\cdot}02$. The result for pure nickel was obtained by T. R. Anantharaman in preliminary measurements, and is probably just significant. The general shape of the curve is very similar to that found by Warren and Warekois for copper–zinc alloys, and there is a steep rise in the concentration of stacking fault near the critical composition where the free energies of the two phases become equal at room temperature. A similar high value of

Fig. 1



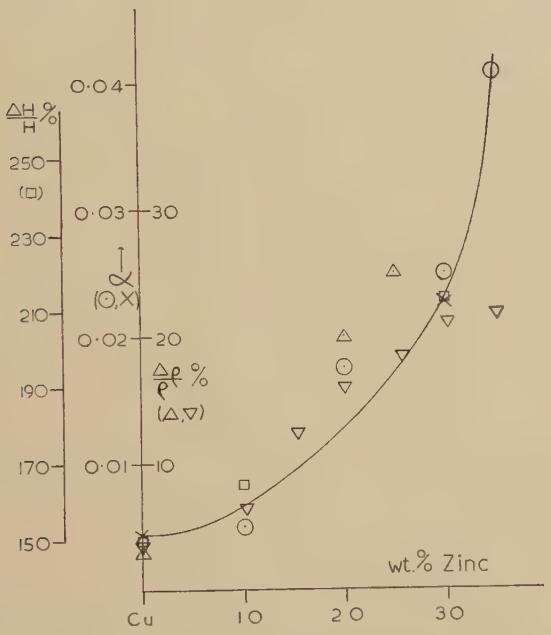
Stacking Faults and Changes in Electrical Resistance in Cobalt–Nickel Alloys.
× Measured Stacking Fault Parameter.
△ Percentage Resistance Change after 35% Reduction in Area (Broom and Barrett 1953).

$\alpha(0{\cdot}024)$ was obtained from filings of a cobalt–iron alloy containing 7·9% iron, which is near the critical composition in this system. As a check on the methods, we measured α in a copper-30% zinc alloy, obtaining a result (0·023) in very good agreement with that of the American workers. Very careful measurements were made on heavily cold-worked pure copper filings, but the shifts in peak positions were so small that only the order of magntiude of α could be determined. Our value of 0·004 is thus compatible with that of 0·007 obtained by Greenough and Smith (1955). It should be noted that the diffraction theory is based on the assumption that faults form on one set of planes only, whereas in the present experiments they are formed equally on all sets of {111} planes. To a first approximation, however, the concentration of faults on each set of planes may be taken to be $\alpha/4$.

Broom and Barrett (1953) found that the increase in electrical resistivity caused by cold-working cobalt–nickel alloys had a maximum in the region where heavy faulting is expected (fig. 1). This result had been

predicted by Broom (1952 a) who first suggested that the large increases in resistance found on cold-working certain face-centred cubic alloys might be due to the formation of stacking faults. Such faults on one set of planes constitute a formidable obstacle to slip on intersecting planes, and in fact the alloys concerned are those which show markedly unequal hardening of latent and active slip planes. Recent work on alpha-brass (Piercy *et al.* 1955) has shown that there is a hardening of the latent planes, as required by this explanation, rather than a softening of the active planes. The connection between percentage increases in resistivity and hardness at room temperature and the formation of stacking faults is also evident for the copper–zinc alloys, as shown in fig. 2.

Fig. 2



Stacking Faults and Changes in Electrical Resistance and Hardness in Copper–Zinc Alloys.

○ Measured Stacking Fault Parameter (Warren and Warekois 1955).
× Measured Stacking Fault Parameter (Present Work).
△ Percentage Resistance Increase for Log Strain 2·3 (Boas and Nicholas 1953).
▽ Percentage Resistance Increase for 84% Reduction in Area (Crampton, *et al.* 1941).
□ Percentage Increase in Hardness after 90% Reduction by Rolling (Copper Development Association).

The measurements are, of course, made at temperatures where there are still some point defects (probably single vacancies) contributing to the resistance, but it seems likely that stacking faults are responsible for the

rapid variation with composition. A survey of the literature suggests that whenever there is a tendency to have low fault energy (as shown by transformations to hexagonal phases, or by prolific formation of annealing twins), the increase of resistance on cold-working and the rate of work-hardening are unusually high.

Since $\frac{1}{2}a\ \langle 110 \rangle$ dislocations are dissociated into narrow ribbons of stacking fault on {111} planes, it is tempting to equate the measured α values with the fractional areas of the close-packed planes contained within these extended dislocations. Any such interpretation is, however, highly speculative, since the diffraction theory is based on the assumption that faults occupy whole planes, and the diffraction effects of narrow ribbons of stacking fault cannot be predicted in detail at present. If the separation of the partial dislocations is nevertheless calculated on this basis, the results obtained are in good agreement with theoretical estimates of the separations (Seeger and Schoeck 1953, Seeger 1955 a). A density of 10^{12} dislocations cm^{-2} corresponds to separations of $\sim$8 atoms for $\alpha=0{\cdot}004$ (the lower limit observable) and $\sim$80 atoms for $\alpha=0{\cdot}040$ (the highest value yet observed). The actual variation in the width of an extended dislocation must be less than indicated by the α values, however, since the dislocation density characteristic of the heavily cold-worked state also varies with composition (Smallman, private communication). The present position thus appears to be that the diffraction effects are in accordance with Paterson's theory, but the physical significance of the faulting parameter is uncertain. An alternative possibility is that large faults are produced irreversibly during the deformation by processes such as the breaking of Cottrell–Lomer locks in the way suggested by Friedel (1955).

Our results for the cobalt–nickel alloys, and those of the Americans for copper–zinc alloys, show that the measurable stacking faults are removed gradually as the annealing temperature is raised. Typical results from one of our alloys were that about 20% of the measurable faults were removed by a short anneal at 270°C, and 60% at 400°C; longer anneals at these temperatures produced no further changes. The faulting parameter became too small to be detected in all alloys after a half-hour anneal at 500°C, although appreciable line broadening remained to much higher temperatures. These results seem to be more readily explained if the faults are extended dislocations. Reduction in the numbers of faults would then occur both by the mutual elimination of opposite dislocations and (possibly) by the aligning of dislocations into sub-boundaries, where their widths would be reduced. There is also some evidence (Tamman 1936, Maddigan and Blank 1940) that in alpha–brass, a major part of the extra electrical resistance anneals out in the temperature range where the faults diminish.

Seeger (1955 a, b) has emphasized that materials with very narrow dislocations are expected to have plastic properties quite different from those with wide extended dislocations. The known differences in the

Suggested Correlation of Stacking Fault Energies in F.C.C. Materials and Increases in Electrical Resistivity caused by Deformation

Group	Material	Deformation	Increase in resistivity, %	Reference
I	Al	log strain 2·3, D	0·5	Boas and Nicholas (1953).
II	Cu, OFHC	84%, D	3	Crampton, *et al.* (1941).
	Cu, spec. pure	log strain 2·3, D	3	Boas and Nicholas (1953).
	Cu	log strain 2·3, D	3	Boas and Nicholas 1953.
	Cu	40–80%, D	2	Takahasi (1930).
	Ni	99%, D	8	Geiss and van Liempt (1927).
	Ni	log strain 2·3, D	6	Boas and Nicholas (1953).
	Ni	90%, D	4	Broom (1952 a).
	Ag	98%, D	5	Tamman and Dreyer (1933).
	Ag	60%, D	3	Takahasi (1930).
	Pt	99%, D	6	Geiss and van Liempt (1927).
	Pd	96%, D	4	Tamman and Dreyer (1933).
	Cu–Ni (various compn.)	84%, D	4	Crampton *et al.* (1941).
	Ag–25% Au	97%, R	1	Tamman and Dreyer (1933).
	Ag–35% Au	90%, D	1	Broom (1952 a).
III	Ag–30%Cd	97%, R	28	Tamman and Dreyer (1933).
	Ag–23%Zn	97%, R	19	Tamman and Dreyer (1933).
	Cu–20%Zn	log strain 2·3, D	20	Boas and Nicholas (1953).
	Cu–25%Zn	log strain 2·3, D	25	Boas and Nicholas (1953).
	Cu–30%Zn	84%, D	26	Crampton *et al.* (1941).
	Cu–5·7%Al	log strain 2·3, D	28	Boas and Nicholas (1953).
	Cu–6% Al	log strain 1·96, D	27	Broom and Clothier (1952).
	Cu–7·5%Al	84%, D	33	Crampton *et al.* (1941).
	Co–31%Ni	35%, D	8	Broom and Barrett (1953).

In the deformation column, the percentages give the reduction in area caused by drawing (D) or in thickness by rolling (R).

Group I. Fault energy >100 ergs cm^{-2}, dislocation widths $\sim$1 atoms.

Group II. Fault energy $\sim$40 ergs cm^{-2}, dislocation widths $\sim$10 atoms. Faults in cold-worked material just detectable by x-rays.

Group III. Fault energy <10 ergs cm^{-2}, dislocation widths $\sim$40 atoms. Faults in cold-worked material readily detectable.

behaviour of (say) copper and brass, together with the data on resistivity and work-hardening, suggest that it may be more convenient to subdivide face-centred cubic materials into three ranges of stacking fault energy. Such a classification is shown in the table, but it must be emphasized that the alloys in Group III are in critical composition ranges, and there may be a gradual transition to Group II behaviour as either the composition or the temperature is altered. Assuming that point defects (when present) and narrow dislocations give increases in resistivity of only a few per cent, it is seen that the magnitude of the resistance change can serve as a rough guide to the stacking fault energy. There is also some evidence (Broom and Clothier 1952, Broom 1952 b) that the anisotropy of the stacking fault resistance can be detected for Group III materials.

Quantitative measurements on stacking faults are necessarily made on powders, in order to obtain high deformations, and also because line shifts in polycrystalline bulk specimens may be caused by residual intergranular stresses (see e.g. Greenough 1952). However, faulting in sufficient amounts may be detected from streaks on single crystal oscillation photographs, and it is clear that the role of stacking faults in deformation can now best be clarified by experiments on single crystals given controlled deformations. Low temperature experiments should also be interesting, both because the amount of faulting should increase, and because the critical alloys of low fault energy are probably the most favourable materials for the production of face-centred cubic mechanical twins, as observed in copper by Blewitt *et al.* (1955).

Acknowledgments

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Stark-Splitting in Crystals

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Rare-earth ions in crystals have absorption and fluoresence spectra consisting of line multiplets separated by a wave number difference $\sim$1000 cm^{-1}. The lines within a group may have separations $\sim$100 cm^{-1} and are all due to transitions to a multiplet of levels with the same total angular momentum (J). The splitting of the levels within a multiplet has been examined by Bethe (1929) using the methods of group theory, and his treatment of the crystallographic double-groups needed to deal with half integral values of J has been extended by Opechowski (1940) to include tetrahedral and rhombohedral symmetries. Using a slightly different approach Hellwege (1948) has considered in detail the wave functions involved in a discussion of the levels in fields having the symmetry of any of the 32 crystallographic point groups. These theoretical papers do not express the results in a condensed form suitable for comparison with experiment ; but Gobrecht (1937) compiled a table of the number of levels for a given J in fields of different symmetry, and this table has been frequently reproduced (Freed 1942 ; Yost *et al.* 1947 ; Pringsheim 1949). Unfortunately all these tabulations are in error in their treatment of fields of tetragonal symmetry which are asserted to produce as many distinct levels as the lowest symmetry groups. It is now intended to rectify this error which is not present in the original work, and to discuss the splitting of the levels in nearly cubic fields, which frequently occur both in pure salts, e.g. alums, and when impurity ions requiring charge compensation enter cubic crystals (Runciman 1955).

The 32 crystallographic point groups can be classified into seven systems and for the present purpose these will be grouped under four headings (*a*) cubic, (*b*) hexagonal, containing the hexagonal and rhombohedral systems, (*c*) tetragonal, and (*d*) lower symmetry including orthorhombic,

monoclinic and triclinic. The point groups in Schoenflies notation under these four headings are as follows :

(*a*) Cubic :—O_h, O, T_d, T_h, T.

(*b*) Hexagonal :—D_{6h}, D_6, C_{6v}, C_{6h}, C_6, D_{3h}, C_{3h}, D_{3d}, D_3, C_{3v}, S_6, C_3.

(*c*) Tetragonal :—D_{4h}, D_4, C_{4v}, C_{4h}, C_4, D_{2d}, S_4.

(*d*) Lower symmetry :—D_{2h}, D_2, C_{2v}, C_{2h}, C_2, C_s, S_2, C_1.

A recalculation of the splitting for integral J has shown that all point groups within one of these four categories produce the same splitting, and the number of lines for given J is listed in table 1. The number of triply, doubly and singly degenerate levels (n_3, n_2, n_1) are written below the total number of levels for cubic symmetry. An inspection of correlation tables for the species of a group and its subgroups (Wilson, Decius and Cross 1955) shows that for a cubic field with slight hexagonal distortion the triply degenerate levels will split into a doublet and a singlet, and the doublet levels will remain doublet ; whereas for a tetragonal distortion the triplet levels will again split into a doublet and a singlet, and doublet levels will split into two singlets.

Table 1. Term-splitting for Integral J

J	0	1	2	3	4	5	6	7	8	General rule
Cubic	1	1	2	3	4	4	6	6	7	add 5 every 6
(n_3 n_2 n_1)	(001)	(100)	(110)	(201)	(211)	(301)	(312)	(411)	(421)	add (311) every 6
Hexagonal	1	2	3	5	6	7	9	10	11	add 4 every 3
Tetragonal	1	2	4	5	7	8	10	11	13	add 3 every 2
Lower symmetry	1	3	5	7	9	11	13	15	17	$2J+1$

Table 2. Term-splitting for Half-integral J

J	1/2	3/2	5/2	7/2	9/2	11/2	13/2	15/2	17/2	General rule
Cubic	1	1	2	3	3	4	5	5	6	add 2 every 3
All other groups	1	2	3	4	5	6	7	8	9	$J+\frac{1}{2}$

Levels are counted as doublets, even though the characters for the representations are distinct, so long as the characters are complex conjugates of each other, as in this case there is no frequency separation of the levels. For orthorhombic and lower symmetry there is complete splitting into $2J+1$ levels for integral J.

For half-integral J, i.e. for an odd number of electrons, the calculation of the splitting involves the double-groups. Bethe's results have been confirmed and table 2 lists the number of levels, either quadruplets or doublets, for any of the five cubic point groups. For all other groups

there is total splitting into $J+\frac{1}{2}$ doublets, all levels remaining doubly-degenerate in any electric field (Kramers 1930). It therefore appears as if ions with an even number of electrons are better suited to determine the site-symmetry of salts or impurity-activated solids containing rare-earth ions. The application of a magnetic field, which in general removes all degeneracy, may also help in finding the terms responsible for an emission or absorption line.

ACKNOWLEDGMENT

I wish to thank Dr. D. F. Johnston of the Theoretical Physics Division at Harwell for many invaluable discussions.

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CIX. REVIEWS OF BOOKS

Expanding Universes. By Professor E. Schrödinger. (Cambridge University Press, 1956.) [93 pp.] Price 17*s.* 6*d.*

Professor Schrödinger has produced a most interesting and valuable little book. Although the title obviously derives from the modern astronomical problem of the universe as a whole, the author is not here concerned with this actual problem but only with the purely geometrical aspects of conceptual expanding models irrespective of whether their metrics in fact satisfy the field equations of general relativity. In other words the book is an exercise in multi-dimensional geometry aimed at clarifying those parts relevant to expanding universes generally, and anyone wishing to sharpen his wits in this intricate realm will find an opportunity in following the quickly moving arguments.

Beginning with the properties of flat and spherical expanding models, the paths of particles and light rays, and the corresponding frequency changes, are investigated for them. The properties of waves in general Riemannian space-time and in an expanding universe are studied by means of Hamilton–Jacobi theory. The book will help to illumine the theory of the actual expanding universe in much the way that a treatise on conics might have helped in the understanding of Keplerian orbits three centuries ago.

The subject matter of Professor Schrödinger's book, which is based on a brief course of lectures, is so sublime that the most reverent of styles would be appropriate to it, but regrettably the book is marred by rather too many distracting informalities and oddities of phrasing, and other such imperfections that could easily have been eliminated with great advantage to an otherwise elegant and handsome contribution to a highly important and recondite subject. R. A. L.

The Interference Systems of Crossed Diffraction Gratings: The Theory of Moiré Fringes. By J. Guild. [Pp. vi+152.] (Oxford: Clarendon Press, 1956.) Price 25*s.*

This monograph by Mr. J. Guild of the Light Division of the National Physical Laboratory is one of those rare gems—an entirely original book, which breaks completely new ground. In this text Mr. Guild develops the theory of the production of so called Moiré fringes arising from the crossing of line gratings. In particular he discusses the properties, localizations and general characteristics of these hitherto almost neglected optical effects. As he points out, Moiré fringes were first briefly discussed in 1874 by Lord Rayleigh, but the subject has remained since then in abeyance. It is only very recently that the National Physical Laboratory has vigorously begun to study these fringes with the double purpose of (*a*) studying the characteristics of the gratings themselves and (*b*) using the Moiré fringes for precision metrological measurement. Mr. Guild has supplied a vital part of these new developments.

This is a small book and highly specialized with six chapters as follows: (1) Introduction (2) Fundamentals (3) Gratings in Series (4) Characteristics of Moiré Fringes (5) Metrological Applications (6) Grating Defects.

It is unfortunate that classical optics in university courses now, of necessity, occupies much less attention than formerly, owing to the pressure of more 'modern' branches of physics. Thus few undergraduates can be expected to profit from this text. On the other hand all postgraduate workers in optics and every teacher of optics will welcome this text with plaudits, for the author treats the subject in a masterly and comprehensive fashion. Most notable is the fact that the author has a sense of the physical feeling for what the light waves are doing and his many original and valuable conclusions reflect this feeling for the physics rather than the mathematics of the processes he is discussing. S. T.

Mesons and Fields. Vol. I (Fields). By S. S. Schweber, H. A. Bethe and F. de Hoffman. Vol. II (Mesons). By H. A. Bethe and F. de Hoffman. (Evanston, Ill.: Row, Peterson & Co.) [Pp. (Vol. I) xvi + 449; (Vol. II) xiii + 446.] Price 63*s.* each volume.

These two volumes give a comprehensive account of the properties of mesons and of the various theoretical methods which are available for their discussion. There are chapters on μ-mesons and on K-mesons, but, naturally enough, the main part of the detailed discussion is about pions.

The first volume deals with what may be called the 'highbrow' forms of theory; i.e. the general and abstract development of quantum field theory, with particular emphasis on the covariant relativistic formulation and the renormalization programme. The first section gives the theory of relativistic wave equations without field quantization, and includes Feynman's form of positron theory. In Feynman's scheme positrons are regarded as electrons travelling backwards in time, and it is possible to draw the well-known Feynman diagrams in which upward pointing lines represent electrons and downward pointing lines represent positrons. The discussion of Feynman kernels and boundary conditions, which is necessary for this scheme, is fairly subtle; it must make heavy reading for any but the specialists.

The central section of Vol. I is a comprehensive account of the methods of field quantization; it includes valuable accounts of Wick's theorem on expressing a product of operators as a sum of creation and annihilation processes, and of the Gupta–Bleuler method for eliminating the longitudinal part of the electromagnetic field. The remainder of Vol. I is mostly concerned with the renormalization procedure. At this point a critical discussion would have been very interesting and very valuable; instead, the authors are content to follow the standard references fairly closely. The renormalization idea, originally due to Kramers, is as follows: subtracting the infinities arising in field theory is equivalent to redefining the mass and charge of the elementary particles. Unfortunately the mathematics of this subtraction scheme becomes very involved: we have to consider expressions which are infinite series of terms, each term being a divergent multiple integral. It requires just a little faith to believe that all the subtractions work out as they should.

The second volume, besides a comprehensive description of the properties of mesons, contains an account of the practical phenomenological form of theory which is used fairly successfully to give a coherent picture of pion phenomena. This material makes Vol. II an extremely valuable reference book for anyone wishing to interpret new results. The simple form of theory which is used is mainly based on the pseudoscalar nature of the pion, in variance in isotopic space and a pseudovector interaction between pions and nucleons: this approach has been successful in explaining a wide variety of scattering and pion production phenomena, in particular by the methods developed by Chew.

In one section of Vol. II the rigorous field theory methods are used to analyse pion–nucleon scattering by the so-called Tamm–Dancoff method. The analysis is complicated, and unfortunately the results are not so satisfactory as those coming from the recent phenomenological theory of Low and Chew which is briefly mentioned in an Appendix. Here we have an example of the odd situation in contemporary field theory. Since 1946 some very high powered mathematical effort has been put into developing the field theory of interacting elementary particles, but, so far as mesons and nucleons are concerned, the practical results are much less than might have been expected. We have certainly learned much about selection rules, but when we have to estimate a cross-section, the phenomenological theory in many cases gives as satisfactory, or more satisfactory, results than the strict application of field theory. This does not mean that field theory can be ignored; it remains the only precise way

we know for describing elementary particles in accord with special relativity, the uncertainty principle and quantum statistics.

Professor Bethe and his collaborators by giving so thorough an account, both of the field theory and the more practical methods, have produced two particularly valuable volumes. J. H.

La Résonance Paramagnétique Nucléaire. Edited by P. Grivet. (Paris : Centre National de la Recherche Scientifique, 1955). [Pp. 198.] Price 1600 *frs.*

Here is a useful book which brings together the contributions to the study of nuclear magnetic resonance by workers in France and in Geneva. Although covering a good deal of ground the book does not attempt a comprehensive survey of the subject, but instead is built mainly round the work of the authors. There are eight authors who contribute chapters on their own specialities. Coming from so many sources it is not surprising that the chapters are not closely integrated and vary in difficulty. Many readers making their first serious acquaintance with the subject through this book may find the early theoretical chapters rather difficult. The printing is in typescript characters and this appears to be the cause of the rather frequent misprints, especially in mathematical formulae.

The Editor, Professor Grivet, sketches the background of the phenomenon in an introductory chapter, and points out some of the salient features. In the second chapter Dr. Gabillard gives a theoretical account from a macroscopic viewpoint. He develops Bloch's equations and discusses their solution under various conditions. In chapter 3, Dr. Ayant looks at the phenomenon from the quantum-mechanical standpoint and calculates the linewidth and relaxation time for liquids and the spectrum for some solids. The theoretical concepts would have been made clearer if these two chapters could have been dovetailed.

In chapters 4 and 5 Dr. Gabillard and Dr. Soutif give an account of methods detecting nuclear magnetic resonance. The methods described were : the crossed-coil induction method, the Q meter method, the bridge method, the spin-echo method, other pulse methods, marginal oscillator and super-regenerative oscillator methods. Some of these methods are analyzed in greater detail and practical circuits are given with values of the required components.

In chapter 6 Dr. Béné, Dr. Denis and Professor Extermann of Geneva discuss methods of producing the steady magnetic field. A surprising amount of space is devoted to the production of the field by iron-free coils. Electromagnets and their stabilization are discussed, but rather little is said about permanent magnets.

In chapter 7 Dr. Béné tabulates nuclear moment data from all sources and Dr. Ayant refers to the shifts and corrections encountered when measuring nuclear magnetic moments by the nuclear magnetic resonance method. Chapter 8 is concerned with pure quadruple resonance, the theory being given by Dr. Ayant and the experimental section, with full circuit details, by Dr. Buyle-Bodin. The book closes with a brief review of the theory of nuclear moments by Béné, Denis and Extermann. E. R. A

Progress in Semiconductors. I. Edited by A. F. Gibson, P. Aigrain and R. E. Burgess. (London : Heywood & Co.) [Pp. 220.] Price 50*s*.

This is a collection of review papers relative to the theory and technology of semiconductor devices. The aim in most cases seems to be bibliographical rather than expository, but the papers will be convenient as temporary stores of information for those active in the field. J. M. Z.

Astronomical Optics and related subjects. Edited by Z. KOPAL. (Amsterdam: North Holland Publishing Company, 1956). [Pp. xv+428.] Price 90*s*.

THE Editor and publishers are to be congratulated on this book, which gives an account of the proceedings of the Symposium held in the University of Manchester in April 1955. 105 guests from five countries attended this unusually lively Symposium. The 46 papers here reproduced or summarized fall under the following heads: 1. Information theory and optics; 2. Optical images and diffraction; 3. Interferometry and coherence problems; 4. Electronic devices in Astronomical Optics; 5. Resolution Problems and Scintillation; 6. Wide-angle optical systems and aspheric surfaces; 7. Filter photography and thin films. Introductory remarks by Professor Kopal and concluding remarks by Dr. J. Rösch (France) complete the volume.

E. H. L.

Elementary Wave Mechanics, 2nd edition. By W. HEITLER. (Oxford: Clarendon Press. London: Cumberlege.) [Pp. viii+193.] Price 18*s*.

THE second edition of Heitler's well known book is very welcome. It is about half as long again as the first edition; as it is addressed primarily to chemists the expansion is mainly in a section on diatomic molecules and on the chemical bond.

N. F. M.

Surveys in Mechanics: the G. I. Taylor 70th Anniversary Volume. Edited by G. K. BATCHELOR and R. M. DAVIES. (Cambridge University Press, 1956.) [Pp. vii+475.] Price 50*s*.

THIS volume, containing ten articles and a biographical note, was prepared to commemorate Sir Geoffrey Taylor's seventieth birthday on the 7th March, 1956. Each article surveys the present position in a particular field of fluid or solid mechanics in which Taylor has worked sometime during the last forty or fifty years.

The contents of the volume are as follows:

G. I. Taylor: A Biographical Note, by R. V. Southwell.
The Mechanics of Quasi-Static Plastic Deformation in Metals, by R. Hill.
Dislocations in Crystalline Solids, by N. F. Mott.
Stress Waves in Solids, by R. M. Davies.
Rotating Fluids, by H. B. Squire.
The Mechanics of Drops and Bubbles, by W. R. Lane and H. L. Green.
Wave Generation by Wind, by F. Ursell.
Viscosity Effects in Sound Waves of Finite Amplitude, by M. J. Lighthill.
Turbulent Diffusion, by G. K. Batchelor and A. A. Townsend.
Atmospheric Turbulence, by T. H. Ellison.
The Mechanics of Sailing Ships and Yachts, by K. S. M. Davidson.

The articles vary considerably in length from about twenty to over one hundred pages. Most of the surveys give a completely adequate account of the present state of their particular subject. The long article by Lighthill is, however, rather more than a straight survey; it contains some new and interesting material on the properties of shock waves.

The editors and individual authors of this volume are to be congratulated on the production of a work which will undoubtedly help all researchers and teachers of fluid and solid mechanics to keep abreast with some of the subjects which have developed rapidly in recent years. The volume is also a most appropriate tribute to Sir Geoffrey's contributions to research in these fields during the last few decades.

J. W. M.

Solid State Physics. Volume II. Edited by F. Seitz and D. Turnbull. (London: Academic Books Ltd.) [Pp. xii+468.] Price $10.

The second volume of this ambitious series is as welcome and useful as the first. It contains articles on nuclear magnetic resonance by Pake and by Knight, on neutron diffraction by Shull and on specific heats by de Launay. Finally there is a massive and comprehensive article by Seitz and Koehler on the important subject of displacement of atoms during irradiation which in one hundred and fifty pages gives a most detailed survey of this subject. N.F.M.

Valency and Molecular Structure. By E. Cartmell and G. W. A. Fowles. (London: Butterworths Scientific Publications.) [Pp. xi+256.] Price 32*s.* 6*d.*

This book is intended as an introduction to valency theory and molecular structure for students beginning an honours course in chemistry. It begins with a clear and simple outline of the experimental foundations of the quantum theory and its application to isolated atoms, and then goes on to discuss the valence-bond and molecular orbital methods for the application of the theory to molecules. The discussion contains little mathematical treatment, and will undoubtedly be very useful to the many students who find difficulty in mastering the elements of theoretical chemistry.

The second half of the book deals with the nature of the bonding in solid structures, the structures of simple inorganic substances, complex compounds, and electron-deficient molecules. In this section the scope of the work is much more ambitious, and the structures of the most simple and many complex inorganic compounds are described and interpreted in terms of electronic configurations, special attention being paid to substances of current interest, such as ferrocene, the higher hydrides of boron, interhalogen compounds, metal carbonyls, and metal alkyls. It is a pity, however, that the discussion is restricted throughout to molecular configuration and dimensions; the energies and restoring force constants of covalent bonds provide important information which should take its place with the bond lengths to which the authors (and chemists in general) pay so much attention. Even when dealing with interatomic distances, the treatment is hardly critical enough; the structure of nitric oxide, for example, is discussed on the basis of 'calculated' N=O and N—O bond lengths with no indication that 'calculations' of bond lengths in heteronuclear inorganic molecules rest at the present time on intelligent but empirical correlations of observed bond distance and inferred bond order. These defects detract somewhat from the value of an otherwise useful introduction to structural chemistry.

The book is well produced, except for the printing in diagrams of structures; this is sometimes almost invisible. A. G. S.

Radiation Dosimetry. Edited by G. J. Hine and G. L. Brownell. (New York: Academic Press Inc.) [Pp. xiv+932.] Price $22.

This book has eighteen chapters each on a different topic and each contributed by a recognised authority in the field. It is divided into three main sections; the first entitled "Fundamental Principles of Dosimetry" gives an excellent and detailed account of the physical principles involved in the interaction of radiation with matter and the application of these principles to dosimetry. The second section entitled "Radiation Detectors and their Calibration" surveys the various means of quantitatively measuring radiation and includes an excellent and up to date chapter on "Calorimetric Methods"—particularly pertinent in the light of the recent recommendations of the International Commission on Radiological Protection that radiation dose should be measured

as energy absorbed per gm of absorber. The third section—" Radiation Fields and their Dosimetry "—covers the whole field of dose measurement for electromagnetic radiation with quantum energies from 200 kev to 30 Mev and in addition there are chapters on high energy electrons, heavy charged particles, and neutrons used as external sources and another on the dosimetry of internally administered isotopes.

The references, given at the end of each chapter, are excellent and up to date.

This book gives an account of a rapidly expanding subject and collects in one volume material which previously could be consulted only in research papers and separate books.

It can be highly recommended for all workers in the field. C. L. S.

BOOK NOTICES

Peaceful Uses of Atomic Energy. Proceedings of the International Conference in Geneva, August 1955. A United Nations Publication. Available from H.M. Stationery Office, P.O. Box 569, London, S.E.1. and other branches.

Volume 2. " Physics ; Research Reactors." [Pp. 471.] Price 57*s*.
Volume 3. " Power Reactors." [Pp. 389.] Price 54*s*.
Volume 4. " Cross Sections important to Reactor Design." [Pp. viii+357.] Price 54*s*.
Volume 5. " Physics of Reactor Design." [Pp. viii+545.] Price 63*s*.
Volume 7. " Nuclear Chemistry and Effects of Irradiation." [Pp. 691.] Price 70*s*.
Volume 8. " Production technology of the materials used for nuclear energy." [Pp. ix+627.] Price 70*s*.
Volume 9. " Reactor Technology and Chemical Processing." [Pp. x+771.] Price 70*s*.
Volume 11. " Biological Effects of Radiation." [Pp. 399.] Price 57*s*.
Volume 14. " General Aspects of the Use of Radioactive Isotopes: Dosimetry." [Pp. 305.] Price 45*s*.
Volume 15. " Applications of Radioactive Isotopes and Fission Products in Research and Industry." [Pp. viii+327.] Price 54*s*.
Volume 16. " Record of the Conference." [Pp. x+203.] Price 36*s*.

Reactor Handbook. Produced by the United States Atomic Energy Commision. (London and New York : McGraw-Hill.)

" Research Reactors ". [Pp. 422.] Price $6.50.
" Reactor Handbook : Physics ". [Pp. 709.] Price $12.
" Reactor Handbook : Engineering ". [Pp. 1075.] Price $15.
" Reactor Handbook : Materials ". [Pp. 610.] Price $10.50.
" Neutron Cross Sections ". [Pp. 328.] Price $12.
" Chemical Processing and Equipment ". [Pp. 302.] Price. $6.

Information Theory. Papers read at a symposium on Information Theory held at the Royal Institution, London, September 1955. (London : Butterworths ; New York : Academic Press Inc.) [Pp. xii+401.] Price 70*s*.

The Medical Effects of the Atomic Bomb in Japan. Edited by A. W. Oughterson and S. Warren. (London : McGraw-Hill.) [Pp. 477.] Price 60*s*.

Engineering Fluid Mechanics. By Charles Jaeger. (Glasgow: Blackie & Son Ltd.) [Pp. xviii+529.] Price 60*s*.

The Fundamental Study of Mathematics. By Sotaro Nitto. (Tokyo: Maruzen Co. Ltd.) [Pp. 83.] Price not stated.

The Diamond Tool Industry in 1955. Issued by the Industrial Diamond Information Bureau, 32–34 Holborn Viaduct, London, E.C.1. (pp. 12.) Free of charge.

As in previous years, the Industrial Diamond Information Bureau has compiled a survey of the diamond tool industry, dealing with various scientific and technical aspects. The subject-matter, mainly based on the Bibliography of Industrial Diamond Applications, is sub-divided into 16 main groups; about 200 references are quoted. The literature mentioned can be made available to enquirers.

Spheroidal Wave Functions. By J. A. Stratton, P. M. Morse, L. J. Chus, J. D. C. Little and F. J. Carbato. (London: Chapman & Hall.) [Pp. xi+613.] Price 100*s*.

Christian Theology and Natural Science. By E. L. Mascall. (London: Longmans, Green and Co. Ltd.) [Pp. xxi+323.] Price 25*s*.

The Theory of Photons and Electrons. By J. M. Jauch and F. Rohrlich. (London: Academic Books Ltd.) [Pp. ix+488.] Price $10.

Mathematical Theory of Elasticity, second edition. By I. S. Sokolnikoff. (London: McGraw–Hill.) [Pp. xi+476.] Price 71*s*. 6*d*.

Functional Analysis. By F. Riesz and B. Sz.-Nagy. (Glasgow: Blackie and Son Limited.) [Pp. xii+468.] 65*s*.

Microscopic Petrography. By E. W. Heinrich. (London: McGraw–Hill.) [Pp. 296.] Price 49*s*.

Kernmomente. By H. Kopfermann. (Frankfurt: Akademische Verlagsgesellshaft.) [Pp. xvi+462.] Price DM54.

Rectangular-Polar Conversion Tables. By E. H. Neville. (Cambridge University Press.) [Pp. xxix+109.] Price 30*s*.

ERRATA

The Publishers regret that in the October number of the Philosophical Magazine a transposition of subject-titles occurred.

On p. 967 the title should read "Energy Levels of ^{40}A" and not "Periodicity in Ice-Flowers".

On p. 975 the title should read "Periodicity in Ice-Flowers" and not "Energy Levels of ^{40}A".

The Publishers tender their apologies to the Authors concerned and trust that no inconvenience has been caused to them.

[The Editors do not hold themselves responsible for the views expressed by their correspondents.]